Pair Production and Annihilation

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Relative answers!

All of the questions on this sheet were supposed to be answered ignoring any effects due to relativity. However, one of the speeds calculated as an answer was greater than the speed of light, which is clearly unphysical. For the interested, here the effects of special relativity are taken into account. Einstein discovered in 1905 that the mass m of a body is not constant, but increases with velocity, according to

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}},$$

where the 'rest mass' m_0 represents the mass of the body when it is not moving, and c is the speed of light, equal to $3 \times 10^8\,\mathrm{m\,s^{-1}}$. As Richard Feynman said:

> For those who want to learn just enough about it so they can solve problems, that is all there is to the theory of relativity—it just changes Newton's laws by introducing a correction factor to the mass.

1. (a) *no difference*, as $m = m_0$ when the particles are at rest and $v = 0$.

   (b) At $0.1c$, the effects of relativity are slight. Without special relativity, we used Newtonian physics to work out the energy of the incoming electron as the sum of the energy bound up in its rest mass and its kinetic energy:

$$E_\mathrm{e} = m_\mathrm{e}c^2 + \frac{1}{2}m_\mathrm{e}v^2.$$

   In special relativity, the energy of a body is given by:

$$\boxed{E = \sqrt{m_0^2c^4 + p^2c^2},}$$

   where p is the (relativistic) momentum of the body[1]. In Newtonian physics, we know that $p = mv$, and relativistically, we can write

$$p = \frac{m_0 v}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

[1] Here I'm ignoring the fact that momentum is a vector quantity (though really these are vector equations, which of course work equally well as scalar identities where e.g. $p^2 = |\mathbf{p}|^2$).

Substituting this into the equation above (after squaring both sides),

$$
\begin{aligned}
E^2 &= m_0^2c^4 + p^2c^2 \\
&= m_0^2c^4 + \frac{m_0^2v^2c^2}{1-\frac{v^2}{c^2}} \\
&= \frac{m_0^2c^4\left(1-\frac{v^2}{c^2}\right) + m_0^2v^2c^2}{1-\frac{v^2}{c^2}} = \frac{m_0^2c^4 - m_0^2v^2c^2 + m_0^2v^2c^2}{1-\frac{v^2}{c^2}} \\
&= \frac{m_0^2c^4}{1-\frac{v^2}{c^2}}, \text{ so} \\
E &= \frac{m_0c^2}{\sqrt{1-\frac{v^2}{c^2}}}.
\end{aligned}
$$

This expression still gives a frequency of 1.24×10^{20} Hz for the γ photons, and the (wrong) non-relativistic formula gives an error of <0.1%.

(c) At $0.5c$, the non-relativistic error is more serious, at ~3%. The non-relativistic approximation gives a frequency of 1.388×10^{20} Hz, the relativistic formula,

$$E_\gamma = \frac{m_e c^2}{h}\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}} = 1.424 \times 10^{20}\,\text{Hz}.$$

2. (a) Conservation of energy still applies, but using the relativistic expression for the kaon energies:

$$
\begin{aligned}
2m_p c^2 &= \frac{2m_{K^0}c^2}{\sqrt{1-\frac{v^2}{c^2}}} \\
v &= c\left(1-\frac{m_{K^0}^2}{m_p^2}\right)^{\frac{1}{2}}.
\end{aligned}
$$

(b) Here the non-relativistic treatment actually leads to a calculated speed 57% larger than the relativistic value:

$$
\begin{aligned}
v &= c\left(1-\frac{m_{K^0}^2}{m_p^2}\right)^{\frac{1}{2}} \\
&= 2.54 \times 10^8\,\text{m s}^{-1},
\end{aligned}
$$

which is reassuringly less than the speed of light *in vacuo*.

3. (a) *no difference*

(b) *no difference*

(c) *no difference*

4. The error in this case from using a non-relativistic form for the energy of the protons is 14%.

$$\frac{2m_{\mathrm{p}}c^2}{\sqrt{1-\frac{v^2}{c^2}}} = 2m_{(\Sigma+)}c^2$$

$$v = c\left(1-\frac{m_{\mathrm{p}}^2}{m_{(\Sigma+)}^2}\right)^{\frac{1}{2}}$$
$$= 1.63\times 10^8\,\mathrm{m\,s^{-1}}.$$

From the problems on this sheet, it can be seen that the errors introduced by ignoring the effects of relativity are very small at low speeds, but grow dramatically larger as the speed becomes near to that of light. This is shown in the following graph which compares the relativistic and non-relativistic forms for particle energies, which are respectively

$$E = \sqrt{m_0^2c^4 + p^2c^2}$$

and

$$E = m_0c^2 + \frac{p^2}{2m_0}.$$

Total energy of a moving particle