Pair Production and Annihilation

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mass of electron $= 9.1 \times 10^{-31}\,\mathrm{kg}$, mass of proton $= 1.67 \times 10^{-27}\,\mathrm{kg}$, $h = 6.64 \times 10^{-34}\,\mathrm{J\,s}$.

1. An electron and positron annihilate producing two gamma ray photons. Calculate the frequency of the photons if

   (a) the electron/positron pair are stationary,
   For the photons, $E_\gamma = hf_\gamma$, and the conversion of the e^- e^+ rest masses into energy is governed by $E = mc^2$. We can therefore write:

$$\begin{aligned} 2hf_\gamma &= 2m_\mathrm{e}c^2 \\ f_\gamma &= \frac{m_\mathrm{e}c^2}{h} \\ &= \frac{9.1 \times 10^{-31}\,\mathrm{kg} \times (3 \times 10^8\,\mathrm{m\,s^{-1}})^2}{6.64 \times 10^{-34}\,\mathrm{J\,s}} \\ &= 1.23 \times 10^{20}\,\frac{\cancel{\mathrm{kg}}\,\cancel{\mathrm{m}^2}}{\cancel{\mathrm{s}^2}\,\cancel{\mathrm{J}}\,\mathrm{s}}, \text{ since } \mathrm{J} \equiv \mathrm{kg\,m^2\,s^{-2}} \\ &= 1.23 \times 10^{20}\,\mathrm{Hz}. \end{aligned}$$

   (b) the electron/positron pair are each travelling at $0.1c$,
   This time, the energy has another term, from the initial kinetic energy of the e^- and e^+. Since we are told to ignore relativity, we can use $E_k = \frac{1}{2}mv^2$, giving:

$$\begin{aligned} 2hf_\gamma &= 2m_\mathrm{e}c^2 + 2\left(\frac{1}{2}m_\mathrm{e}v^2\right) \\ f_\gamma &= \frac{2m_\mathrm{e}c^2 + m_\mathrm{e}v^2}{2h} \\ &= \frac{2m_\mathrm{e}c^2 + m_\mathrm{e}(0.1c)^2}{2h} \\ &= \frac{2m_\mathrm{e}c^2 + 0.01m_\mathrm{e}c^2}{2h} \\ &= \frac{2.01m_\mathrm{e}c^2}{2h} \\ &= 1.24 \times 10^{20}\,\mathrm{Hz}, \text{ i.e. a factor } \frac{2.01}{2} \text{ of the previous answer.} \end{aligned}$$

   (c) the electron/positron pair are each travelling at $0.5c$, By the same method,

$$\begin{aligned} f_\gamma &= \frac{2m_e c^2 + m_e v^2}{2h} \\ &= \frac{2m_e c^2 + m_e (0.5c)^2}{2h} \\ &= \frac{2.25 m_e c^2}{2h} \\ &= 1.39 \times 10^{20}\,\text{Hz}. \end{aligned}$$

where c = speed of light $= 3 \times 10^8\,\text{m}\,\text{s}^{-1}$.

[Ignore any effects due to relativity.]

2. A proton and antiproton, each travelling at a negligible speed, collide and annihilate according to the following reaction: $\text{p}\overline{\text{p}} \rightarrow \text{K}^0\overline{\text{K}}^0$. Assuming that the kaons move off at the same speed,

   (a) by conservation of energy show that the speed of the kaons can be found by

   $$v = \left(\frac{2c^2}{m_{\text{K}^0}} (m_\text{p} - m_{\text{K}^0}) \right)^{\frac{1}{2}},$$

   where m_{K^0} is the mass of the kaon, m_p is the mass of the proton, and c is the speed of light.
   The energy released in the annihilation of the p and $\overline{\text{p}}$ (which comes from their rest energies, as their speed is negligible) is turned into the particle–antiparticle pair created in this interaction, and any left over goes into the kinetic energy of these new particles. Energy is conserved, meaning the energy beforehand must be the same as the energy afterwards, so we can write:

   $$\begin{aligned} 2m_\text{p}c^2 &= 2m_{\text{K}^0}c^2 + 2\left(\frac{1}{2} m_{\text{K}^0} v^2\right) \\ 2m_\text{p}c^2 &= 2m_{\text{K}^0}c^2 + m_{\text{K}^0} v^2 \\ m_{\text{K}^0} v^2 &= 2m_\text{p}c^2 - 2m_{\text{K}^0}c^2 \\ v^2 &= \frac{2c^2\left(m_\text{p} - m_{\text{K}^0}\right)}{m_{\text{K}^0}} \\ v &= \left(\frac{2c^2}{m_{\text{K}^0}} (m_\text{p} - m_{\text{K}^0}) \right)^{\frac{1}{2}}. \end{aligned}$$

   (b) given the K^0 has a mass of 8.9×10^{-28} kg, calculate the speed of the kaons.

$$
\begin{aligned}
v &= \left(\frac{2c^2}{m_{\mathrm{K}^0}}(m_{\mathrm{p}} - m_{\mathrm{K}^0})\right)^{\frac{1}{2}} \\
&= \left(\frac{2 \times (3 \times 10^8\,\mathrm{m\,s^{-1}})^2}{8.9 \times 10^{-28}\,\mathrm{kg}} \times (1.67 \times 10^{-27}\,\mathrm{kg} - 8.9 \times 10^{-28}\,\mathrm{kg})\right)^{\frac{1}{2}} \\
&= \left(\frac{2 \times (3 \times 10^8\,\mathrm{m\,s^{-1}})^2 \times 7.8 \times 10^{-28}\,\cancel{\mathrm{kg}}}{8.9 \times 10^{-28}\,\cancel{\mathrm{kg}}}\right)^{\frac{1}{2}} \\
&= \left(1.58 \times 10^{17}\,\mathrm{m^2\,s^{-2}}\right)^{\frac{1}{2}} \\
&= 3.97 \times 10^8\,\mathrm{m\ s^{-1}}.
\end{aligned}
$$

NB This answer is utterly unphysical, since we know that the kaons cannot be travelling faster than c. After checking the calculation, there is no error in the logic presented above. The mistake is that this reasoning starts from false premises: the kaons would in fact travel at a speed ($0.846c = 2.54 \times 10^8\,\mathrm{m\,s^{-1}} < c$) where special relativity cannot be neglected, so their mass will be observed to be greater than their rest mass, i.e. the formula $E_k = \frac{1}{2}mv^2$ is not a good enough approximation to model their behaviour (as it would be for low speeds where $v \ll c$).

3. Gamma ray photons can cause pair production, such as the production of an electron–positron pair, according to $\gamma \to \mathrm{e^-e^+}$.

   (a) What is the minimum energy of a gamma ray photon needed to produce this reaction?

$$
\begin{aligned}
E_\gamma &= 2m_{\mathrm{e}}c^2 \\
&= 2 \times 9.1 \times 10^{-31}\,\mathrm{kg} \times (3 \times 10^8\,\mathrm{m\,s^{-1}})^2 \\
&= 1.64 \times 10^{-13}\,\mathrm{J}.
\end{aligned}
$$

   (b) What is the frequency of this photon?

$$
\begin{aligned}
E_\gamma &= hf_\gamma \\
f_\gamma &= \frac{E_\gamma}{h} = \frac{1.64 \times 10^{-13}\,\cancel{\mathrm{J}}}{6.64 \times 10^{-34}\,\cancel{\mathrm{J}}\mathrm{s}} \\
&= 2.47 \times 10^{20}\,\mathrm{Hz}.
\end{aligned}
$$

   (c) The gamma ray has to interact with another object, such as a nucleus. Why is this so?
   The other object (which must have mass) needs to recoil so that momentum and energy are conserved in the interaction.

4. A proton–antiproton pair may interact according to $\mathrm{p\overline{p}} \to \Sigma^+\overline{\Sigma}^+$ if the protons are given enough energy.
   Calculate the minimum velocity of the protons for this reaction to occur.
   [mass of $\Sigma^+ = 1.99 \times 10^{-27}\,\mathrm{kg}$.]

$$
\begin{aligned}
2m_{\mathrm{p}}c^2 + 2\left(\frac{1}{2}m_{\mathrm{p}}v^2\right) &= 2m_{(\Sigma^+)}c^2 \\
v &= \left(\frac{2c^2}{m_{\mathrm{p}}}\left(m_{(\Sigma^+)} - m_{\mathrm{p}}\right)\right)^{\frac{1}{2}} \\
&= \left(\frac{2 \times \left(3 \times 10^8\,\mathrm{m\,s^{-1}}\right)^2}{1.67 \times 10^{-27}\,\cancel{\mathrm{kg}}} \times \left(1.99 \times 10^{-27} - 1.67 \times 10^{-27}\right)\cancel{\mathrm{kg}}\right)^{\frac{1}{2}} \\
&= 1.86 \times 10^8\,\mathrm{m\,s^{-1}}.
\end{aligned}
$$