Scalars and vectors

A.C. Norman, Bishop Heber High School

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In physics we meet various quantities, and we are familiar that some things like mass, energy, charge, length or temperature can be represented by a unit scaled by a single number. Such quantities are known as *scalars*, and these can be handled mathematically in the same way as simple numbers (like the number of potatoes in a sack).

Other quantities in physics have both a size and a direction. A good example is force, which is a push or a pull in a particular direction. Saying 'the overall force on the object is $20\,\mathrm{N}$' is no use in determining what will happen to it unless the direction is specified. Direction is included in the concept of a force, and quantities like this are called *vectors*.

It may seem obvious that all relations between physical quantities, that are the quantitative descriptions of physical process, must be independent of the measuring scale[1] and the frame of reference used.

[1] This is why in all true physics formulas all terms have identical dimensions. For example, in $\mathcal{A} + \mathcal{B} = \mathcal{C}$, while $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ may be absurdly complicated, they must all have the same dimensions. This allows the equation to be written in the *dimensionless* form $\frac{\mathcal{A}}{\mathcal{A}} + \frac{\mathcal{B}}{\mathcal{A}} = \frac{\mathcal{C}}{\mathcal{A}}$—the same method allows turns any valid equation into a dimensionless form—and showing that it is independent of the particular choice of measurement units (just like any equation describing the world should be!)

Vectors are often represented as an arrow with its size and direction representing the size and direction of the quantity. Why can we represent force, say, by an arrow? Because it has the same mathematical transformation properties as an arrow in space. We can represent it in a diagram as if it were an arrow, using a scale such that one unit of force, or one newton, corresponds to a certain convenient length. Once we have done this, all forces can be represented as lengths, because an equation like

$$\mathbf{F} = k\mathbf{r},$$

where k is some constant, is a perfectly legitimate equation. Thus we can always represent forces by arrows, which is very convenient. An equation like the one above will be true in any coordinate system if it is true in one.

The fact that a physical relationship can be expressed as a vector equation assures us that the relationship is unchanged by a mere rotation or translation of the coordinate system. That is the reason why vectors are so useful in physics.

Writing vectors

There are a number of conventions for writing vectors. In some books, you will see vectors in bold type ($\mathbf{r}$), in others they are written with arrows over the letter ($\overrightarrow{r}$). Sometimes you might also see underlining ($\underset{\sim}{r}$), which is much used in handwritten work.

Vector components

Physical phenomena take place in the 3D world around us. Most often, we use a 3D coordinate system to specify positions in space, and other vector quantities. The vector can be described by the coordinate system. Figure 1 shows a position vector that might represent your final position if you started at the origin and walked $4\,\mathrm{m}$ along the x axis, $2\,\mathrm{m}$ in the direction of the z axis (parallel to it) and the climbed a ladder so you were $3\,\mathrm{m}$ above the ground.

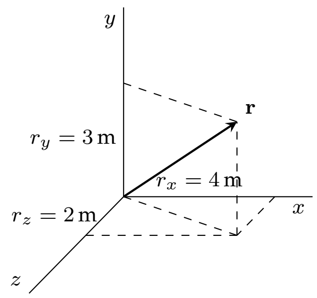

Figure 1: A position vector **r** in a right-handed 3D coordinate system. In some books, the x axis points out, the y axis points to the right and the z axis points up, but since we are also going to use a 2D coordinate system with y up, it makes sense to always have the y axis pointing up

Your new position relative to the origin is a vector that can be written

$$\mathbf{r} = (4, 3, 2)$$
$$x \text{ component } r_x = 4\,\mathrm{m}$$
$$y \text{ component } r_y = 3\,\mathrm{m}$$
$$z \text{ component } r_z = 2\,\mathrm{m}$$

The vector **r** can be represented by three numbers: r_x, r_y and r_z. Each of the numbers is called a *component* of the vector. The x component of the vector **r** is r_x (in this case $4\,\mathrm{m}$). A component is not a vector (which needs three numbers to decribe it in 3D space) since it is only one number, perhaps with a unit.[2] Notice that although the vector is in a sense made up from its components r_x, r_y and r_z, it is not really only *those* particular numbers, since if we were to rotate the axes used to measure them, the three components would change, but the vector **r** would stay the same. The symbol **r** for a vector is a representation of the vector itself, so it will represent the same thing no matter how we turn the axes.

[2] Although it is not a vector, it isn't a scalar either, as an important property of a true scalar is that it doesn't change if we orient the xyz axes differently. Rotating the axes doesn't change an object's mass or temperature, but it will change the x component of its velocity as the the x axis now points in a new direction

Length (magnitude) of a vector

The length of a vector is found by using Pythagoras' theorem (or its extension to higher numbers of dimensions). For the vector **r** from figure 1, the distance you moved from the starting point is given by

$$\sqrt{(4\,\mathrm{m})^2 + (3\,\mathrm{m})^2 + (2\,\mathrm{m})^2} = 5.4\,\mathrm{m}.$$

We say that the *magnitude* $|\mathbf{r}|$ of the vector **r** is $5.4\,\mathrm{m}$.

The magnitude of a vector can be calculated by taking the square root of the sum of the squares of its components. If the vector $\mathbf{r} = (r_x, r_y, r_z)$, then

$$|\mathbf{r}| = \sqrt{r_x^2 + r_y^2 + r_z^2} \text{ (a scalar)}.$$

The magnitude of a vector is always a positive value.[3]

[3] The length of a vector found in this way is a true scalar, as it doesn't depend if the axes are rotated (although the individual components will change, the length of the arrow is invariant).

Multiplying a vector by a scalar

Multiplication by a scalar 'scales' a vector, keeping its direction the same but making its length larger or smaller. Multiplying by a negative scalar reverses the direction of the vector.

Adding vectors

The *resultant* or *vector sum* of two displacement vectors is the result of performing first one and then the other displacement.[4] If two forces act on a body then the resultant force acting on the body is the vector sum of the two. Adding vectors in this way only makes physical sense if they are the same kind of vector, for example both forces acting in three dimensions.

[4] The magnitude of the resulting vector is in general *not* equal to the sum of the magnitudes of the two original vectors!

The sum of two vectors can be interpreted geometrically: to combine two vectors $\mathbf{a}$ and $\mathbf{b}$, all we have to do is to put the head of $\mathbf{a}$ against the tail of $\mathbf{b}$ (without changing the direction of either one), and draw the final arrow from the tail of $\mathbf{a}$ to the head of $\mathbf{b}$. That's all there is to it.

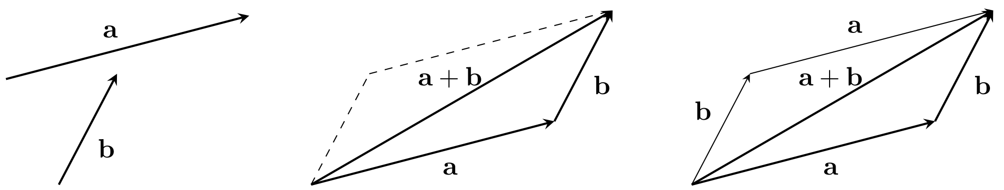

Figure 2: Adding two vectors by placing them head-to-tail. This is sometimes called the *parallelogram rule*, because the resultant vector forms the diagonal of a parallelogram constructed with the vectors along two adjacent sides.

Notice that we can add vectors in any order (mathematicians would say that addition of vectors is commutative), i.e.

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}.$$

Also, any number of vectors can be added like this, with each vector telling you how far, and in in what direction, to move (this is shown in figure 3). The final resultant vector tells you what *single* move to make to end up in the same place.

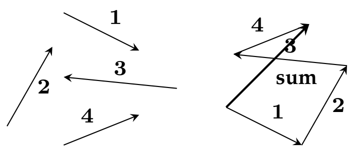

Figure 3: Any number of vectors can be added together in the manner shown in figure 2.

If we have the components of a vector—this is why describing vectors as a set of components is so useful—the addition of the vectors can be done by simply adding their components, i.e.

$$\mathbf{a} + \mathbf{b} = (a_x, a_y, a_z) + (b_x, b_y, b_z) = (a_x + b_x, a_y + b_y, a_z + b_z),$$

and their difference by subtracting them,

$$\mathbf{a} - \mathbf{b} = (a_x, a_y, a_z) - (b_x, b_y, b_z) = (a_x - b_x, a_y - b_y, a_z - b_z).$$

Resolving vectors

Just as two vectors can be added together into one resultant vector, in reverse any vector can be *resolved* into components. To do this, we choose three vectors which do not all lie in the plane to use as a *basis* to use to describe other vectors in terms of their components. Most often we choose to use basis vectors that are all mutually perpendicular to each other (this mathematical property is called orthogonality, the generalization of perpendicularity to a higher number of dimensions), but this is not necessary, and this is shown in 2D for general basis vectors in figure 4.

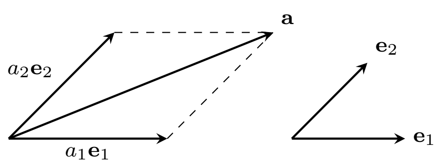

Figure 4: In two dimensions, given two different vectors, $\mathbf{e}_1$ and $\mathbf{e}_2$, it is possible to write any other 2D vector in terms of them:

$$\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2.$$

The two vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ are said to form a *basis* (for the 2D space), whilst the numbers a_1 and a_2 are the components of $\mathbf{a}$ with respect to this basis. We say that the vector has been resolved into components.

To find the components of a vector, we could use a scale drawing, but it is also possible to use trigonometry to calculate the components. This is particularly straightforward when the basis vectors are orthogonal. Consider the force vector **F** in the xy plane shown in figure 5. We should like to represent **F** as (F_x, F_y) by using its components F_x and F_y.

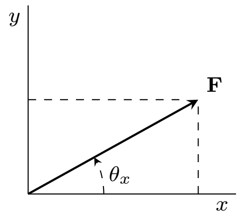

Figure 5: A force vector **F** shown in the xy plane of an orthogonal basis, at an angle θ_x round from the x axis direction.

From the definition of the cosine of an angle we know that

$$\cos\theta_x = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{F_x}{|\mathbf{F}|},$$

and similarly

$$\cos\theta_y = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{F_y}{|\mathbf{F}|},$$

where θ_y is the angle of the vector between the vector **F** and the y axis (equal to $90° - \theta_x$ as shown in figure 6).

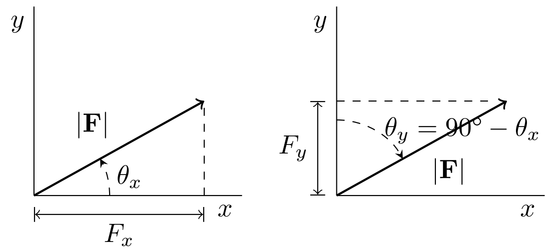

Figure 6: Resolution of a vector **F** into its components F_x and F_y.

The components F_x and F_y of the vector are given by

$$F_x = |\mathbf{F}|\cos\theta_x$$

and

$$F_y = |\mathbf{F}|\cos\theta_y$$

respectively.[5] The force **F** shown in figures 5 and 6 lies between the x and y axes (i.e. $\theta_x < 90°$), but this method works for larger angles as well (for angles betwee $90°$ and $180°$, the cosine function is negative, corresponding to F_x being negative).

[5] You may have noticed that the y component of the vector can also be calculated as $|\mathbf{F}|\sin\theta_x$, and it is often useful to recognize that a vector component can be obtained using sine instead of cosine. There is, however, some advantage of always using calculating in terms of cosines. The method always works, including with non-orthogonal basis vectors, and in 3D, and it avoids having to decide whether to use a sine or a cosine. Just use the cosine of the angle between the direction you are resolving into and the vector.