On density

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Warm-up problems

1. Write down the equation used to calculate density, two units density can be measured in, and show how to convert between them.

$$\text{density} = \frac{\text{mass}}{\text{volume}}, \quad \rho = \frac{m}{V}.$$

Density has the dimension $[\mathrm{M}][\mathrm{L}]^{-3}$, so two units density could be measured in are $\mathrm{g\,cm^{-3}}$ and $\mathrm{kg\,m^{-3}}$. To convert between them:

$$\begin{aligned} 1\,\mathrm{kg\,m^{-3}} &= \frac{1000\,\mathrm{g}}{(100\,\mathrm{cm})^3} \\ &= \frac{1000\,\mathrm{g}}{1\,000\,000\,\mathrm{cm^3}} \\ &= \frac{1}{1000}\mathrm{g/cm^{-3}}, \end{aligned}$$

or

$$1\,\mathrm{g\,cm^{-3}} = 1000\,\mathrm{kg\,m^{-3}}.$$

2. Put the following substances in order of increasing density: water, xenon, iron, cork, olive oil, air, uranium.
air, xenon, cork, olive oil, water, iron, uranium

3. Write down three contexts in which measurements of density can be useful from class.
Finding the concentration of alcohol in beer, checking whether a ship has been overloaded (Plimsoll line), discovering a new element (neon) in air.

Regular problems

4. (a) A piece of anthracite had a volumne of $15\,\mathrm{cm^3}$ and a mass of $27\,\mathrm{g}$. What is its density
   i. in $\mathrm{g\,cm^{-3}}$,

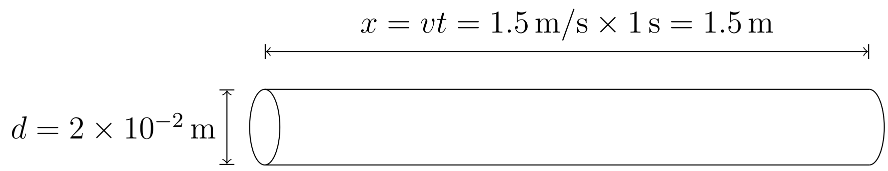
$$\begin{aligned} \rho &= \frac{m}{V} \\ &= \frac{27\,\mathrm{g}}{15\,\mathrm{cm^3}} \\ &= 1.8\,\mathrm{g\,cm^{-3}} \end{aligned}$$

ii. in $\mathrm{kg\,m^{-3}}$?
$1800\,\mathrm{kg\,m^{-3}}$

(b) What is the volume of 1.0 t of sand of density $2.6 \times 10^3\,\mathrm{kg\,m^{-3}}$?

$$\begin{aligned} V &= \frac{m}{\rho} \\ &= \frac{1.0 \times 10^3\,\mathrm{kg}}{2.6 \times 10^3\,\mathrm{kg\,m^{-3}}} \\ &= 0.38\,\mathrm{m^3} \end{aligned}$$

5. Water starts to fill a paddling pool by flowing out of a hosepipe of inner diameter 2 cm at a rate of $1.5\,\mathrm{m\ s^{-1}}$. What mass of water is there in the paddling pool after 1 minute? ($\rho_{\mathrm{water}} = 1000\,\mathrm{kg\ m^{-3}}$)
Consider the amount of water flowing out of the hosepipe each second:



The amount of water that has flowed out in 1 s is

$$\Delta V = \pi \left(\frac{d}{2}\right)^2 x$$

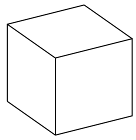
The rate of water flowing out of the hose is the volume per time,

$$\begin{aligned} \frac{\mathrm{d}V}{\mathrm{d}t} &= \pi \left(\frac{d}{2}\right)^2 \frac{\mathrm{d}x}{\mathrm{d}t}, \\ &= \pi \left(\frac{d}{2}\right)^2 v, \end{aligned}$$

note that the expression on the right is a volume per second. The rate of change of mass in the paddling pool is found using $m = \rho V$ and hence the mass of water in the paddling pool after a certain time can be obtained by multiplying by the time:

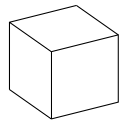
$$\begin{aligned} m &= \rho \frac{\mathrm{d}V}{\mathrm{d}t} t, \\ &= \rho\pi \left(\frac{d}{2}\right)^2 vt, \\ &= 1000\,\mathrm{kg\,m^{-3}} \times \pi \times \left(\frac{2 \times 10^{-2}\,\mathrm{m}}{2}\right)^2 \times 1.5\,\mathrm{m\,s^{-1}} \times 60\,\mathrm{s}, \\ &= 28\,\mathrm{kg}. \end{aligned}$$

6. A light alloy consists of 70% Al and 30% Mg by mass. What would you expect its density to be? ($\rho_{\mathrm{Al}} = 2.7 \times 10^3$ kg m^{-3}, $\rho_{\mathrm{Mg}} = 1.74 \times 10^3$ kg m^{-3})
Imagine making this alloy by starting with large blocks, 700 kg of Al and 300 kg of Mg respectively.



700 kg Al

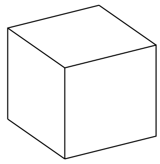
$$V_{\mathrm{Al}} = \frac{m_{\mathrm{Al}}}{\rho_{\mathrm{Al}}}$$



300 kg Mg

$$V_{\mathrm{Mg}} = \frac{m_{\mathrm{Mg}}}{\rho_{\mathrm{Mg}}}$$

These blocks are then melted down and mixed together to form a larger block of the alloy (assuming there is no change in volume in the alloying process).



1000 kg alloy, 70% Al and 30% Mg by mass

The density of the alloy is therefore given by:

$$\begin{aligned}
\rho &= \frac{m}{V}, \\
&= \frac{m}{\frac{m_{\mathrm{Al}}}{\rho_{\mathrm{Al}}} + \frac{m_{\mathrm{Mg}}}{\rho_{\mathrm{Mg}}}}, \\
&= \frac{\rho_{\mathrm{Al}}\rho_{\mathrm{Mg}} m}{\rho_{\mathrm{Mg}} m_{\mathrm{Al}} + \rho_{\mathrm{Al}} m_{\mathrm{Mg}}}, \\
&= \frac{\rho_{\mathrm{Al}}\rho_{\mathrm{Mg}}}{\rho_{\mathrm{Mg}} f_{\mathrm{Al}} + \rho_{\mathrm{Al}} f_{\mathrm{Mg}}},
\end{aligned}$$

where f_{Al} and f_{Mg} are the fractions of Al and Mg, 0.7 and 0.3 respectively. Using this formula gives:

$$\begin{aligned}
\rho &= \frac{\rho_{\mathrm{Al}}\rho_{\mathrm{Mg}}}{f_{\mathrm{Al}}\rho_{\mathrm{Mg}} + f_{\mathrm{Mg}}\rho_{\mathrm{Al}}}, \\
&= \frac{2.7 \times 10^3\,\mathrm{kgm^{-3}} \times 1.74 \times 10^3\,\mathrm{kgm^{-3}}}{0.7 \times 1.74 \times 10^3\,\mathrm{kgm^{-3}} + 0.3 \times 2.7 \times 10^3\,\mathrm{kgm^{-3}}}, \\
&= 2300\,\mathrm{kg\,m^{-3}}.
\end{aligned}$$

7. Explain why solid objects made from some substances float on water and some sink:

   (a) using the idea of density to give a general rule of thumb for substances,
   Solid objects made from substances with $\rho < \rho_{\mathrm{water}}$ will float, whereas solid objects made from substances with $\rho > \rho_{\mathrm{water}}$ will sink.

(b) using the idea of upthrust force (Archimedes' principle) on an object,
Archimedes' principle states that all objects experience an upthrust force (equal to the weight of the fluid displaced) when they are immersed in a fluid.

If the mass of water displaced by an object of volume V is greater than the mass m of the object itself, the upthrust force U when it is fully immersed will be greater than the weight W of the object, and therefore the object will rise above the surface (float) until the weight of the water displaced is equal to the mass of the object. You can easily see that this gives the 'rule of thumb' based on the object's density ρ relative to that of water:

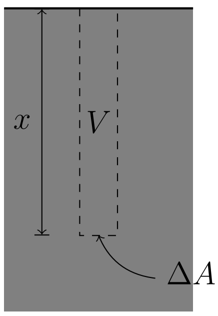
$$\begin{aligned} U &> W \\ m_{\text{water}} g &> mg \\ m_{\text{water}} &> m \\ \rho_{\text{water}} V &> \rho V \\ \rho &< \rho_{\text{water}} \end{aligned}$$

If the mass of water displaced by an object is less than the mass of the object itself, the upthrust will be less than it's weight force, and it will sink (probably until it hits the bottom). NB You can apply the same mathematical argument as above, with the inequality sign reversed!

(c) using the idea of pressure forces on an object.
If a solid object is fully immersed in a fluid such as water, all its external surfaces will experience the pressure of the fluid, which will cause a force F pushing inwards on each part of the surface. A small section of area δA will have a force acting on it according to the equation

$$F = P\delta A.$$

The pressure in a fluid is not uniform. In fact, it varies with depth. It can easily be seen that the pressure at a point at depth x below the surface of the fluid must be exactly enough so that it provides a force to hold up the weight of the volume of fluid V above it which has base area ΔA:



The upward force $F = PA$ on the base of V must balance its weight $W = mg$:

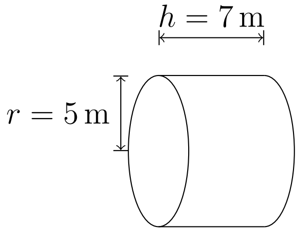
$$\begin{aligned} F &= W \\ PA &= mg \\ PA &= \rho V g \\ PA &= \rho A x g \\ P &= \rho g x \end{aligned}$$

Once it is seen that the pressure in a fluid arises owing to the weight of the fluid itself (the pressure at a point is the result of the weight of the fluid above it) it can be appreciated that forces of upthrust on objects (which are equal to the weight of the fluid displaced, according to Archimedes' principle) arise from the overall effect of the varying pressure forces on different points on the object.

Extension problems

8. The average wind speed at 25.0 m above the Earth's surface, on top of a hill, is $7.0\,\mathrm{m\,s^{-1}}$. A wind turbine on the top of a hill has the rotational centre of its blades at 25.0 m above the ground and has blades of length 5 m. The density of the air is $1.3\,\mathrm{kg\,m^{-3}}$.

   (a) Imagine the circle described by the tips of the rotating blades. What is the volume of air passing through this circle every second?



$$
\begin{aligned}
V &= \pi r^2 h \\
&= \pi \times (5\,\mathrm{m})^2 \times 7\,\mathrm{m} \\
&= 550\,\mathrm{m^3}.
\end{aligned}
$$

   (b) What is the mass of this air?

$$
\begin{aligned}
m &= \rho V \\
&= 1.3\,\mathrm{kg\,m^{-3}} \times 5\,\mathrm{m^3} \\
&= 710\,\mathrm{kg}.
\end{aligned}
$$

   (c) What is the kinetic energy of this mass of air?

$$
\begin{aligned}
\text{K.E.} &= \frac{1}{2}mv^2 \\
&= \frac{1}{2} \times 710\,\mathrm{kg} \times (7\,\mathrm{m\,s^{-1}})^2 \\
&= 17\,500\,\mathrm{J}.
\end{aligned}
$$

   (d) If the turbine converts 40% of the wind energy into electrical energy, what is the power output of the generator?
   The turbine will extract $0.4 \times 17\,500\,\mathrm{J}$ of energy from the air each second, so its power is 7.0 kW.

[British Physics Olympiad Physics Challenge 2004 q.14, cf. AQA PHYA2 Jan 2010 q.7]

9. A votive crown made by command of king Hierto II of Syracuse was suspected of having been fraudulently made in part from silver substituted in place of the gold supplied by the king. Archimedes realized that by comparing the density of the crown to the densities of silver and gold, which are $10.50\,\mathrm{g\,cm^{-3}}$ and $18.90\,\mathrm{g\,cm^{-3}}$ respectively, he could calculate

the composition of the crown.[1] If Archimedes found that the crown weighed 3.52 kg in air and 3.31 g in water, find the composition, by mass, of the alloy, assuming that there has been no volume change in the process of producing the alloy.

[Duncan, Physics: *A textbook for advanced level students*, p.205 q.3, adapted]

The crown's weight seems less in water owing to the force of upthrust. This is (by Archmedes' principle) equal to the weight of water displaced. The mass difference recorded is equal to the mass of water displaced, and allows the volume of the crown to be determined and hence, since we already know its weight, its density.

$$\rho = \frac{m}{V}$$
$$\rho = \frac{m\rho_{\text{water}}}{m - m_{\text{water}}}$$

Now we have density of the crown, we need to work backwards to find out what the proportions of gold and silver are. We have an expression for the density of an alloy of A and B from an earlier question:

$$\begin{aligned} \rho &= \frac{\rho_{\text{A}}\rho_{\text{B}}}{f_{\text{A}}\rho_{\text{B}} + f_{\text{B}}\rho_{\text{A}}}, \\ f_{\text{A}}\rho_{\text{B}} + f_{\text{B}}\rho_{\text{A}} &= \frac{\rho_{\text{A}}\rho_{\text{B}}}{\rho} \\ f_{\text{A}}\rho_{\text{B}} + (1 - f_{\text{A}})\,\rho_{\text{A}} &= \frac{\rho_{\text{A}}\rho_{\text{B}}}{\rho} \\ f_{\text{A}}\rho_{\text{B}} + \rho_{\text{A}} - f_{\text{A}}\rho_{\text{A}} &= \frac{\rho_{\text{A}}\rho_{\text{B}}}{\rho} \\ f_{\text{A}}\rho_{\text{B}} - f_{\text{A}}\rho_{\text{A}} &= \frac{\rho_{\text{A}}\rho_{\text{B}}}{\rho} - \rho_{\text{A}} \\ f_{\text{A}}\,(\rho_{\text{B}} - \rho_{\text{A}}) &= \frac{\rho_{\text{A}}\,(\rho_{\text{B}} - \rho)}{\rho} \\ f_{\text{A}} &= \frac{\rho_{\text{A}}\,(\rho_{\text{B}} - \rho)}{\rho\,(\rho_{\text{B}} - \rho_{\text{A}})} \end{aligned}$$

This gives $f_{gold} = 0.84$ (and hence $f_{silver} = 0.16$), so the crown is made of 84% gold and 16% silver by mass.

[1] This story is told by Vitruvius, a roman architect in the first century BC, who tells us that Archimedes got the idea from taking a bath which overflowed when he immmersed his body. He thereupon is reported to have leapt out of the bath in joy, and, returning home naked, cried out with a loud voice that he had found what he was searching for, exclaiming, in Greek $\varepsilon\nu\rho\eta\kappa\alpha$, (I have found it out). [Vitruvius, *De Architectura*, Book IX, paragraphs 9–12] It is unlikely that Archimedes in fact used this method. Galileo pointed out that it is far more likely that Archimedes in fact used some form of balance to determine the composition of the crown, making use of his law of bouyancy and his law of the lever. [Galileo Galilei, *La Bilancetta*, published in *Opera di Galileo Galilei*, ed. Franz Brunetti, 1980]