On resistivity

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Warm-up problems

1. Explain what is meant by the term *electrical resistivity*, and show that its unit is $\Omega\,\mathrm{m}$?

   The resistivity ρ is the material property which tells you how reluctant a substance is to support an electric current. It is defined by the formula

   $$R = \frac{\rho l}{A},$$

   where R is the resistance of a block of a material of uniform cross section A and length l. Rearranging this formula shows us that resistivity is given by

   $$\rho = \frac{AR}{L},$$

   so in SI units (meaning that area is measured in m^2), the unit of resistivity is given by

   $$\frac{\mathrm{m}^2\Omega}{\mathrm{m}} = \Omega\,\mathrm{m}.$$

2. What makes it more useful than *resistance*? *(Hint: use the analogy between* resistance / resistivity *and* mass / density*.)*

   Resistivity is more useful than resistance because it doesn't depend on the particular size or shape of the object, making it a property of the material alone. Just as density is the shape-independent version of mass, and can more usefully be used to compare materials, resistivity is the 'materials' version of resistance, allowing a fair comparison of material properties to be made.

3. Find the length of constantan wire, radius $5.0 \times 10^{-2}\,\mathrm{cm}$, needed to make a $3.0\,\Omega$ resistor. ($\rho = 4.9 \times 10^{-7}\,\Omega\,\mathrm{m}$)

   $$R = \frac{\rho l}{A} \tag{1}$$

   $$l = \frac{AR}{\rho} \tag{2}$$

   $$= \frac{\pi r^2 R}{\rho} \tag{3}$$

   $$= \frac{\pi \times (5.0 \times 10^{-4}\,\mathrm{m})^2 \times 3.0\,\Omega}{4.9 \times 10^{-7}\,\Omega\,\mathrm{m}} \tag{4}$$

   $$= 4.8\,\mathrm{m}. \tag{5}$$

Regular problems

4. A block of carbon, 1.0 cm by 2.0 cm by 5.0 cm, has a resistance of 0.015 Ω between its two smaller faces. What is the resistivity of carbon?

   The smaller faces will be the 1.0 cm × 2.0 cm ends of the block, and this will make the length 5.0 cm.

$$R = \frac{\rho l}{A} \tag{6}$$

$$\rho = \frac{AR}{L} \tag{7}$$

$$= \frac{(1.0 \times 10^{-2}\,\mathrm{m} \times 2.0 \times 10^{-2}\,\mathrm{m}) \times 0.015\,\Omega}{5.0 \times 10^{-2}\,\mathrm{m}} \tag{8}$$

$$= 6 \times 10^{-5}\,\Omega\,\mathrm{m}. \tag{9}$$

5. A piece of wire has a resistance, R. What will be the resistance of a wire of the same material which is three times as long and twice as thick?

   The formula tells us that

$$R \propto \frac{l}{A}, \text{ i.e.}$$

$$R \propto \frac{l}{r^2}$$

   so if l has increased by a factor of 3 and r has increased by a factor of 2, R will have increased by a factor of $\frac{3}{4}$ (so the answer is that the resistance is $\frac{3}{4}R$).

6. 3.0 m of iron wire of uniform diameter 0.80 mm has a potential difference of 1.50 V across its ends. Calculate the current in the wire. ($\rho_{\mathrm{Fe}} = 9.7 \times 10^{-8}\,\Omega\,\mathrm{m}$)

$$I = \frac{V}{R} \tag{10}$$

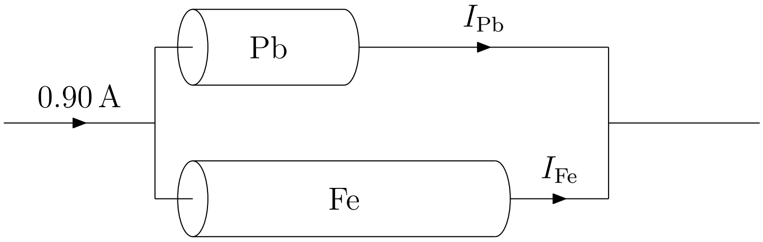
$$= \frac{V\pi r^2}{\rho l} \tag{11}$$

$$= \frac{1.50\,\mathrm{V} \times \pi \times (0.40 \times 10^{-3}\,\mathrm{m})^2}{9.7 \times 10^{-8}\,\Omega\,\mathrm{m} \times 3.0\,\mathrm{m}} \tag{12}$$

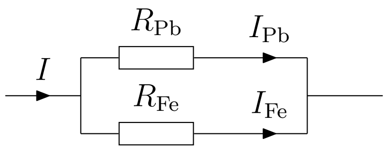
$$= 2.6\,\mathrm{A}. \tag{13}$$

7. A piece of lead wire is connected in parallel with a piece of iron wire of the same diameter but twice as long. If a current of 0.90 A flows through the combination. Find the current in each wire. ($\rho_{\mathrm{Fe}} = 9.7 \times 10^{-8}\,\Omega\,\mathrm{m}$, $\rho_{\mathrm{Pb}} = 2.1 \times 10^{-7}\,\Omega\,\mathrm{m}$).

   Let's start by making sure we understand the problem, by drawing a diagram (and introducing suitable notation I_{Fe} and I_{Pb} for the currents to be found:



We can immediately see that both I_{Fe} and I_{Pb} will be less than $I = 0.90\,\mathrm{A}$, and in fact $I = I_{\mathrm{Fe}} + I_{\mathrm{Pb}}$. The situation we have here is known as a *current divider*:



To calculate the current in one of the branches, we can use Ohm's law for that branch if we know the voltage V across the combination, e.g.

$$I_{\mathrm{Pb}} = \frac{V}{R_{\mathrm{Pb}}}.$$

The voltage across the combination is given (from Ohm's law) by the product of combined resistances in parallel ($R_{\mathrm{Pb}}||R_{\mathrm{Fe}}$) and the current through the combination I:

$$\begin{aligned} V &= I\left(R_{\mathrm{Pb}}||R_{\mathrm{Fe}}\right) && (14) \\ &= \frac{I R_{\mathrm{Pb}} R_{\mathrm{Fe}}}{R_{\mathrm{Pb}} + R_{\mathrm{Fe}}}. && (15) \end{aligned}$$

We can now see the plan of the solution: we can now use the facts that $R = \frac{\rho l}{A}$, the diameters (and therefore the cross sectional areas) of the wires are equal, the iron wire is twice as long as the lead wire, and the resistivity data. Let's carry out the plan to obtain the current in the lead wire:

$$\begin{aligned} I_{\mathrm{Pb}} &= \frac{V}{R_{\mathrm{Pb}}} && (16) \\ &= \frac{I R_{\mathrm{Pb}} R_{\mathrm{Fe}}}{R_{\mathrm{Pb}}\left(R_{\mathrm{Pb}} + R_{\mathrm{Fe}}\right)} && (17) \\ &= \frac{I \rho_{\mathrm{Pb}} l_{\mathrm{Pb}} \rho_{\mathrm{Fe}} l_{\mathrm{Fe}} A_{\mathrm{Pb}}}{A_{\mathrm{Pb}} A_{\mathrm{Fe}} \rho_{\mathrm{Pb}} l_{\mathrm{Pb}} \left(\frac{\rho_{\mathrm{Pb}} l_{\mathrm{Pb}}}{A_{\mathrm{Pb}}} + \frac{\rho_{\mathrm{Fe}} l_{\mathrm{Fe}}}{A_{\mathrm{Fe}}}\right)} && (18) \\ &= \frac{I \rho_{\mathrm{Fe}} l_{\mathrm{Fe}}}{A_{\mathrm{Fe}} \left(\frac{\rho_{\mathrm{Pb}} l_{\mathrm{Pb}}}{A_{\mathrm{Pb}}} + \frac{\rho_{\mathrm{Fe}} l_{\mathrm{Fe}}}{A_{\mathrm{Fe}}}\right)}, \text{ and since } A_{\mathrm{Pb}} = A_{\mathrm{Fe}}, && (19) \\ &= \frac{I \rho_{\mathrm{Fe}} l_{\mathrm{Fe}}}{A_{\mathrm{Fe}} \left(\frac{\rho_{\mathrm{Pb}} l_{\mathrm{Pb}}}{A_{\mathrm{Fe}}} + \frac{\rho_{\mathrm{Fe}} l_{\mathrm{Fe}}}{A_{\mathrm{Fe}}}\right)} && (20) \\ &= \frac{I \rho_{\mathrm{Fe}} l_{\mathrm{Fe}}}{\rho_{\mathrm{Pb}} l_{\mathrm{Pb}} + \rho_{\mathrm{Fe}} l_{\mathrm{Fe}}}. && (21) \\ & && (22) \end{aligned}$$

We can now use the fact that $l_{\text{Fe}} = 2l_{\text{Pb}}$, giving

$$I_{\text{Pb}} = \frac{2I\rho_{\text{Fe}}l_{\text{Pb}}}{\rho_{\text{Pb}}l_{\text{Pb}} + 2\rho_{\text{Fe}}l_{\text{Pb}}} \tag{23}$$

$$= \frac{2I\rho_{\text{Fe}}}{\rho_{\text{Pb}} + 2\rho_{\text{Fe}}} \tag{24}$$

$$= \frac{2 \times 0.90\,\text{A} \times 9.7 \times 10^{-8}\,\Omega\,\text{m}}{2.1 \times 10^{-7}\,\Omega\,\text{m} + 2 \times 9.7 \times 10^{-8}\,\Omega\,\text{m}} \tag{25}$$

$$= 0.43\,\text{A}. \tag{26}$$

Now we can use $I_{\text{Fe}} = I - I_{\text{Pb}} = 0.90\,\text{A} - 0.43\,\text{A} = 0.47\,\text{A}$. We can see that the two currents are fairly similar, implying that the resistances of the two wires are fairly similar. Is this what we expect? $\rho_{\text{Fe}} = 97 \times 10^{-9}\,\Omega\,\text{m}$ is around half of $\rho_{\text{Pb}} = 210 \times 10^{-9}\,\Omega\,\text{m}$), so if the iron wire has double the length, the two wires will indeed have similar resistances (this is not exact: the iron still has a slightly smaller resistance, meaning it gets a slightly larger share of the current (as we have found). This gives us confidence that our solution is correct.

8. In 1881, the ohm was made a base unit (rather than a derived unit like it is today). The 'practical ohm' was defined (at a conference in 1893) as being represented by a column of mercury of cross-section $1\,\text{mm}^2$ at the temperature of melting ice, having length 106.300 cm and mass 14.5421 g.

   (a) What is the density of mercury?
   Note that ρ represents density here:

$$\rho = \frac{m}{V} = \frac{m}{Al} \tag{27}$$

$$= \frac{14.5421\,\text{g}}{1\,\text{mm}^2 \times \frac{1\,\text{cm}^2}{10\,\text{mm} \times 10\,\text{mm}} \times 106.300\,\text{cm}} \tag{28}$$

$$= 13.6802\,\text{g/cm}^3. \tag{29}$$

   Note that 6 s.f. were given in the question, so I have carried this through in the answer. Can I check the answer? Mercury is a metal, so it's much heavier than water ($1\,\text{g/cm}^3$). 10 times heavier? I've sloshed mercury around in a tilt switch before, and I'm mildly surprised, but not shocked, by this value. A jar of mercury is significantly heavier than mercury. For another check, think about barometers: a water barometer is too big for the average room, because atmospheric pressure can support about 10 m of water in a tube with a vacuum at the top. Mercury barometers are used because an atmosphere-supported column of mercury is about 1 m high and is thus room-sized: here is further support for our factor of 10!

   (b) What is its resistivity?
   Back to normal (for this problem sheet): ρ is resistivity:

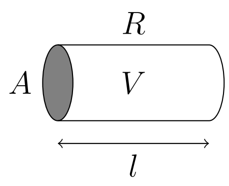
$$\rho = \frac{AR}{l} = \frac{1\,\text{mm}^2 \times \frac{1\,\text{m}^2}{1000\,\text{mm} \times 1000\,\text{mm}} \times 1\,\Omega}{106.300 \times 10^{-2}\,\text{m}} \tag{30}$$

$$= 9.40733 \times 10^{-7}\,\Omega\,\text{m}. \tag{31}$$

   What would you guess for the resistivity of mercury? Better than carbon or lead, but not as good as aluminium or copper? We expect the resistivity of mercury to be comparable to other metals, i.e. of the order $10^{-7}\,\Omega\,\mathrm{m}$, and this seems to agree with our result. The value in Kaye & Laby for 273.2 K is $94.1 \times 10^{-8}\,\Omega\,\mathrm{m}$.

(c) How accurately was the ohm defined at this time? (i.e. 1 part in 100, 1 part in 100 000...)
From the information given in the definition, we can take the uncertainty in length as (106.300 ± 0.001) cm and mass (14.5421 ± 0.0001) g. As percentages, these are 0.0009% and 0.0007% respectively. Whilst defining the substance as mercury and the temperature sets both the density (and therefore the exact cross sectional area) and the resistivity, there may be problems owing to the cross sectional area not being uniform (will this increase, decrease or make no difference to the resistance?), the mercury not being pure, or the temperature not being exact (leading to changes in the density and resistivity), we shall assume that these are negligible.[1] The two percentage uncertainties will add (since l and m are combined by $\times$ / $\div$ in the definition) to give 0.002%, or 1 part in 500.

9. A wire of uniform cross-section has a resistance of R. If it is drawn to three times the length, but the volume remains constant, what will be its resistance?
Let's draw a diagram of the original wire:



For the original wire, $R = \frac{\rho l}{A}$. As the wire is drawn out, the resistivity remains unchanged, so the scaling relation between the resistance R and the new length l and new cross sectional area A is

$$R \propto \frac{l}{A}.$$

Since the volume $V = Al$ remains unchanged, $A \propto \frac{1}{l}$, giving the overall scaling relation

$$R \propto l^2.$$

The scaling relation is shorthand for

$$\frac{R_{\text{drawn wire}}}{R_{\text{original wire}}} = \left(\frac{l_{\text{drawn wire}}}{l_{\text{original wire}}}\right)^2, \tag{32}$$

or

$$R_{\text{drawn wire}} = R_{\text{original wire}} \left(\frac{l_{\text{drawn wire}}}{l_{\text{original wire}}}\right)^2. \tag{33}$$

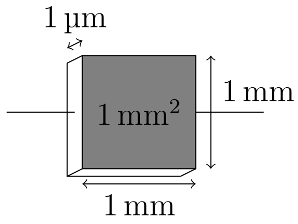
The ratio of the lengths $l_{\text{drawn wire}}/l_{\text{original wire}}$ is 3, so the resistance of the drawn wire will be 9 times the resistance of the original wire.

[1] Although, according to Wikipedia, the mercury column method of realizing a physical standard ohm turned out to be difficult to reproduce, owing to the effects of non-constant cross section of the glass tubing.

Extension problems

10. (from Nelkon & Parker, *Advanced Level Physics*) A thin film resistor in a solid state circuit has a thickness of 1 µm and is made of nichrome of resistivity $10^{-6}\,\Omega\,\mathrm{m}$. Calculate the resistance available between opposite edges of a $1\,\mathrm{mm}^2$ area of film

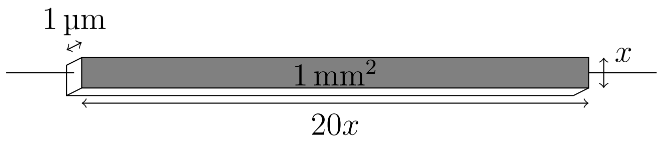
    (a) if it is square shaped,



$$R = \frac{\rho l}{A} \tag{34}$$

$$= \frac{10^{-6}\,\Omega\,\mathrm{m} \times 1 \times 10^{-3}\,\mathrm{m}}{1 \times 10^{-6}\,\mathrm{m} \times 1 \times 10^{-3}\,\mathrm{m}} = 1\,\Omega. \tag{35}$$

    (b) if it is rectangular, 20 times as long as it is wide.



Although the shape has changed, the volume is still $1\,\mathrm{mm}^2 \times 1\,\mu\mathrm{m}$, so it is only the length and cross sectional area that have changed. Sound familiar? We can use the scaling relation from the previous problem

$$R_{\text{rectangular film}} = R_{\text{square film}} \left(\frac{l_{\text{rectangular film}}}{l_{\text{square film}}} \right)^2. \tag{36}$$

The ratio of the lengths is $\sqrt{20}$, so the resistance of the rectangular film will be 20 times the resistance of the original square film (i.e. $20\,\Omega$). Sceptical?

$$20x \times x = 1\,\mathrm{mm}^2 \tag{37}$$

$$x = \sqrt{\frac{1\,\mathrm{mm}^2}{20}} \tag{38}$$

$$= 0.224\,\mathrm{mm}. \tag{39}$$

$$R = \frac{\rho l}{A} \tag{40}$$

$$= \frac{10^{-6}\,\Omega\,\mathrm{m} \times 20 \times 0.224 \times 10^{-3}\,\mathrm{m}}{1 \times 10^{-6}\,\mathrm{m} \times 0.224 \times 10^{-3}\,\mathrm{m}} \tag{41}$$

$$= 1\,\Omega. \tag{42}$$

11. The 1861 ohm was chosen to be $10^9\,\mathrm{ab}\Omega$ [*Nature* Vol. 24, 512 (1881)], in order to be a convenient size, because the abΩ in use at the time was very small. The original abΩ was defined in 1838 for electrical usage from the 'ohmad'. This ohmad was in turn defined as the resistance of one foot of number 11 copper wire (which has diameter 0.0907 inch). A definition based on familiar wires was natural at the time, as the telegraph was critically affected by the resistance of its wires, which set the interval at which repeater stations had to be provided. The resistivity of copper is 6.58×10^{-7} ohm-inch. How big was the original ohmad in modern Ω?

$$R = \frac{\rho l}{A} \tag{43}$$

$$= \frac{6.58 \times 10^{-7}\,\Omega\,\mathrm{inch} \times 12\,\mathrm{inch}}{\pi \times \left(\frac{0.0907\,\mathrm{inch}}{2}\right)^2} \tag{44}$$

$$= 1.22 \times 10^{-3}\,\Omega. \tag{45}$$