On internal resistance

A.C. Norman
ACN.Norman@radley.org.uk

Warm-up problems

1. Explain how the formula

$$\varepsilon = IR + Ir$$

is arrived at, including what the various terms mean. It can sometimes be written as $V = \varepsilon - Ir$. What does the V represent, and what is the combination Ir?

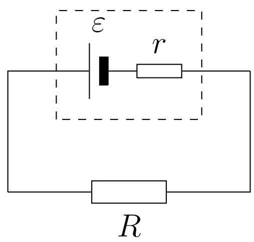
Real world cells have a certain internal resistance r to current flow. When we work out the current I, we need to take this into account as well as the circuit resistance of R, so we find that

$$I = \frac{\varepsilon}{R + r}, \tag{1}$$

where ε is the energy transformed into electrical energy per unit charge, known as the electomotive force (e.m.f.) This equation is often rearranged (as in the question) to $\varepsilon = IR+Ir$, so that each term has the units of energy/charge (in SI units, V). IR is then the work done by the cell per unit charge on the external circuit, and Ir is the energy dissipated per unit charge within the power source (i.e. the work done by the current in the internal resistance).

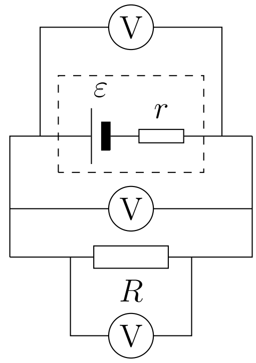
The potential difference across the circuit is IR, and this voltage is sometimes referred to as the *terminal p.d.*, usually given the symbol V, as is is the voltage across the terminals of the cell (not to be confused with *terminal velocity* which is a 'false friend' here as it has a different and unrelated meaning). This terminal p.d. is less than the e.m.f. of the cell, by an amount Ir, sometimes termed the 'lost volts'.

2. Explain the following diagram. What does the dashed box represent, and where might you place a voltmeter to measure the *terminal p.d.*?

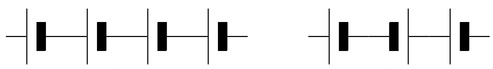


The diagram is a representation of a cell connected to a circuit of resistance R. The dashed box represents the outer case of the cell (within which we have no access), and the situation of internal resistance can be treated *as though* inside the battery there is an ideal cell (with no internal resistance) connected in series with a small resistor with resistance equal to the internal resistance r. Notice that this is just like a power supply of voltage V connected across a potential divider: the two resistances in this case are r and R.

To measure the terminal p.d., a voltmeter can be placed in any of the positions shown below (they are all equivalent as the wires have no resistance and so no p.d. across them):



3. Each of the cells shown below has e.m.f. 1.5 V and internal resistance 0.3 Ω. What is the e.m.f. and internal resistance of the battery combination in each case?



   (a) In the case of 4 cells of the same polarity in series, the total e.m.f. is $4 \times 1.5\,\text{V} = 6.0\,\text{V}$, and the resistances also add in series, to give $4 \times 0.3\,\Omega = 1.2\,\Omega$.

   (b) If there are three cells in series, with two having the same polarity and one in reverse, the e.m.f. (which has polarity) will be $2 \times 1.5\,\text{V} - 1.5\,\text{V} = 1.5\,\text{V}$. The resistances do not have polarity, and so in series they simply add together, giving $3 \times 0.3\,\Omega = 0.9\,\Omega$.

Regular problems

4. (a) Does the internal resistance of a battery increase or decrease its terminal voltage when it supplies current? Justify your answer.
   When current is supplied, some energy per unit charge is dissipated in the internal resistance of the battery, meaning there is less energy available for the circuit (across the terminals of the battery). The terminal voltage therefore goes down as current increases.

   (b) What non-physical (silly) results might a too-simplistic model of a battery, which does not account for the effects of internal resistance, lead to?
   At GCSE, cells are magical devices that always give the same amount of energy to each coulomb of charge as they push it on its way around a circuit. If this were true, we could add more and more branches to a circuit (reducing its overall resistance) and meaning that the current could increase without limit, and thus the power ($P = IV$) drawn from the cell would also be unlimited.

   We know this can't be true, and at A level we learn that the chemicals in a cell which allow it to supply electrical energy also mean that it has a certain internal resistance to current flow.

5. A high resistance voltmeter reads 3.0 V when connected across the terminals of a battery on an open circuit (no other components are connected) and 2.6 V when the battery sends a current of 0.2 A through a lamp. What is

   (a) the e.m.f. of the battery,
   The e.m.f. of the battery is the same as the terminal p.d. when no current is flowing (so no volts are 'lost', Ir, in the internal resistance). Thus the e.m.f. is 3.0 V

   (b) the terminal p.d. of the battery when supplying 0.2 A,
   Again, the high resistance voltmeter tells us the correct answer here, i.e. 2.6 V.

   (c) the 'lost volts',
   3.0 V-2.6 V=0.4 V.

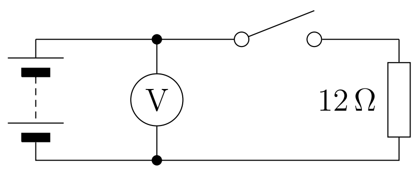
   (d) the internal resistance of the battery,
   The 'lost volts' are equal to Ir, so

$$r = \frac{\text{lost volts}}{I} \tag{2}$$
$$= \frac{0.4\,\text{V}}{0.2\,\text{A}} \tag{3}$$
$$= 2.0\,\Omega. \tag{4}$$

   (e) the resistance of the lamp?
   The terminal p.d. $V = IR$, so

$$r = \frac{V}{I} \tag{5}$$
$$= \frac{2.6\,\text{V}}{0.2\,\text{A}} \tag{6}$$
$$= 13\,\Omega. \tag{7}$$

6. In the following circuit, the voltmeter has a very high resistance. It reads 6.0 V when the switch is open and 4.8 V when it is closed. What is the e.m.f. and internal resistance of the battery?

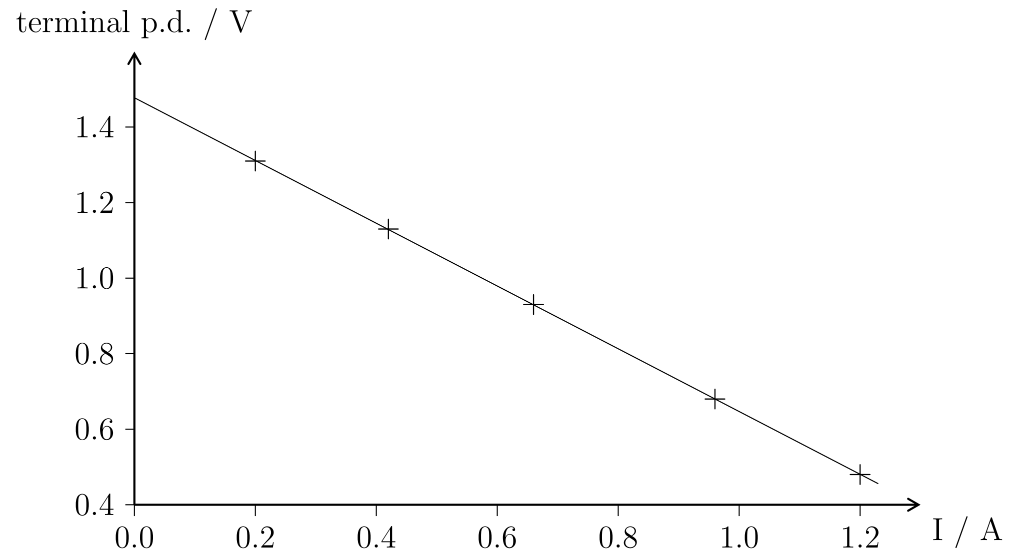


When the switch is open, the high resistance voltmeter will read the e.m.f. of the battery, so $\varepsilon = 6.0\,\text{V}$. When the switch is closed, the voltmeter reads the terminal p.d., IR so the current $I = \frac{V}{R} = \frac{4.8\,\text{V}}{12\,\Omega} = 0.40\,\text{A}$. Now we can find the internal resistance r from knowing the 'lost volts' $Ir = 1.2\,\text{V}$. $r = \frac{1.2\,\text{V}}{0.40\,\text{A}} = 3\,\Omega$.

7. (WJEC PH1 June 2013 Q3) A student carries out an experiment to determine the emf and internal resistance of a cell. The p.d. across the cell is measured when it is supplying various currents. The following readings are obtained.

current / A	0.20	0.42	0.66	0.96	1.20
p.d. / V	1.31	1.13	0.93	0.68	0.48

(a) Plot these results on the grid (p.d. on the y-axis and current on the x-axis) and draw a line through your points.



(b) Use your graph to determine

i. the e.m.f. of the cell,
On the graph, we have V on the y-axis and I on the x-axis, so

$$\underbrace{V}_{y=} = \underbrace{\varepsilon}_{c+} - \underbrace{Ir}_{mx}. \tag{8}$$

The y-intercept is thus the e.m.f., 1.48 V.

ii. the internal resistance of the cell.
In the same way, the (negative of the) gradient is the internal resistance, 0.83 Ω.

(c) The cell is then connected to a torch bulb of resistance 6.0 Ω for 20 minutes. Calculate the charge that flows through the bulb in this time. Assume the e.m.f. remains constant.

$$Q = It \tag{9}$$
$$= \frac{\varepsilon}{R+r}t \tag{10}$$
$$= \frac{1.48\,\mathrm{V}}{6.0\,\Omega + 0.83\,\Omega} \times 20\,\mathrm{minute} \times \frac{60\,\mathrm{s}}{\mathrm{minute}} \tag{11}$$
$$= 260\,\mathrm{C}. \tag{12}$$

8. (WJEC PH1 June 2014 Q3) A torch battery converts 6480 J of chemical energy into electrical energy while supplying a current of 0.15 A for 2 hours. In this time only 5932 J of this energy is supplied to the bulb. Calculate

(a) the charge that flows,

$$Q = It \tag{13}$$
$$= 0.15\,\mathrm{A} \times 2\,\mathrm{hour} \times \frac{60\,\mathrm{minute}}{\mathrm{hour}} \times \frac{60\,\mathrm{s}}{\mathrm{minute}} \tag{14}$$
$$= 1080\,\mathrm{C}. \tag{15}$$

(b) the e.m.f. of the battery
Remembering that the e.m.f. of the battery is defined as the electrical energy evolved in the battery per unit of charge,

$$\varepsilon = \frac{6480\,\mathrm{J}}{1080\,\mathrm{C}} = 6.0\,\mathrm{V}. \tag{16}$$

(c) the potential difference across the bulb,
Remembering that p.d. between two points is defined as the electrical energy transformed per unit of charge flowing between the two points,

$$V = \frac{5932\,\mathrm{J}}{1080\,\mathrm{C}} = 5.5\,\mathrm{V}. \tag{17}$$

(d) the battery's internal resistance.
The 'lost volts' are seen from the answers above to be 0.5 V. This is equal to Ir, and since the current is known, $r = \frac{0.5\,\mathrm{V}}{0.15\,\mathrm{A}} = 3.3\,\Omega$.

Extension problems

9. Prove the *maximum power theorem*, that the maximum power delivered to a circuit by a battery of e.m.f. ε and internal resistance r occurs when the circuit resistance $R = r$. What fraction of the energy supplied by the chemical energy of the battery is transferred to the circuit in this case?

The power delivered to the circuit resistance R is given by

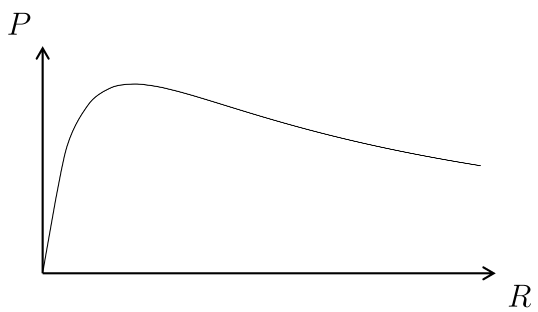
$$P = I^2 R \tag{18}$$

$$= \frac{\varepsilon^2}{(R+r)^2} \times R \tag{19}$$

Examining this function, we can see that when $R \to \infty$, the power delivered to the circuit will also tend to zero. This makes sense, as the current will decrease toward zero.

Although you might think that to get the maximum power, the maximum current is wanted, and therefore the minimum circuit resistance, this expression suggests that when $R = 0$ the power delivered to the circuit will equal zero! (This is because in that short circuit, all the power will be dissipated as heat in the internal resistance of the battery.

There is a maximum somewhere in between these two extremes:



To find this maximum, let us differentiate our expression for P with respect to R and set this gradient of the curve equal to zero in the usual way:

$$P = \frac{\varepsilon^2 R}{(R+r)^2} \tag{20}$$

$$\frac{\mathrm{d}P}{\mathrm{d}R} = \frac{\varepsilon^2}{(R+r)^2} - \frac{2\varepsilon^2 R}{(R+r)^3}. \tag{21}$$

Setting $\frac{\mathrm{d}P}{\mathrm{d}R} = 0$,

$$0 = \frac{\varepsilon^2}{(R+r)^2} - \frac{2\varepsilon^2 R}{(R+r)^3} \tag{22}$$

$$\frac{2\varepsilon^2 R}{(R+r)^3} = \frac{\varepsilon^2}{(R+r)^2} \tag{23}$$

$$\frac{2R}{(R+r)^3} = \frac{1}{(R+r)^2} \tag{24}$$

$$\frac{2R}{R+r} = 1 \tag{25}$$

$$2R = R+r \tag{26}$$

$$R = r. \tag{27}$$

This result—known as the *maximum power theorem*—applies to *any* source of e.m.f. and states that a given source of e.m.f. delivers the maximum amount of power to a load when the resistance of the load is equal to the internal resistance of the source. Note that $R = r$ is the condition that the power delivered by a *given* source is a maximum: if a source with the same e.m.f. but lower internal resistance is used, the power delivered will go up, even though is is not the maximum power it is capable of providing.

The power supplied by the chemical energy of the battery in this case will be $P = I\varepsilon = \frac{\varepsilon^2}{2R}$, and the power dissipated in the circuit is $P = \frac{\varepsilon^2 R}{4R^2} = \frac{\varepsilon^2}{4R}$: so half of the power is dissipated in the battery and only half is transferred to the circuit in this maximum power case!

One more thing. It's always good to generalize a solution, and I offer two clues here. That factor of one half crops up over and over in physics (think about $\frac{1}{2}mv^2$!) And secondly, this maximum power transfer type problem is something to look out for in the future: a good name for an abstraction (a reusable idea) to take forward from this problem is *impedance matching*.