On capacitor charging

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Warm-up problems

1. What is the formula for how the voltage across a capacitor which is being charged varies with time? Be very careful to show what each symbol means.

$$V = V_0 \left(1 - e^{-\frac{t}{RC}}\right),$$

where V is the voltage across the capacitor, V_0 is the final voltage across the capacitor when it is fully charged, t is the time elapsed since charging started (when the capacitor had zero charge), R is the resistance in the circuit, and C is the capacitance of the capacitor

2. Find out what a *supercapacitor* is. What are its advantages over a battery in an application such as electric cars?

A supercapacitor is a capacitor which has a much higher capacitance than the usual capacitors, and can store much more energy per unit volume or mass. It can charge and discharge much faster than a battery, and does not degenerate as quickly when 'cycled' charged and discharged. Supercapacitors may therefore be ideal for electric cars, which need to charge quickly in order to avoid hanging around waiting, charge/discharge many thousands of times, and deliver high current for acceleration.

Regular problems

3. A capacitor of capacitance 200 µF is charged through a resistor of resistance 300 kΩ.

(a) What is the time constant for this circuit?

$$RC = 200\,\mu\text{F} \times 300\,\text{k}\Omega = 60\,\text{s}.$$

(b) How long will the capacitor take to charge to (i) 50%, (ii) 75%, (iii) 90% and (iv) 99.9% of a full charge?

$$Q = Q_0 \left(1 - e^{-t/RC}\right) \tag{1}$$

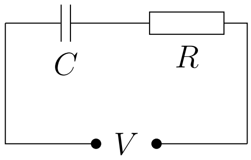
$$1 - \frac{Q}{Q_0} = e^{-t/RC} \tag{2}$$

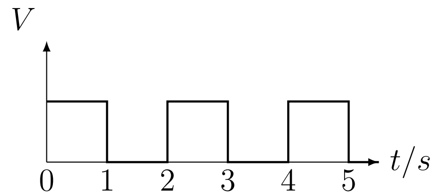
$$\ln\left(1 - \frac{Q}{Q_0}\right) = -\frac{t}{RC} \tag{3}$$

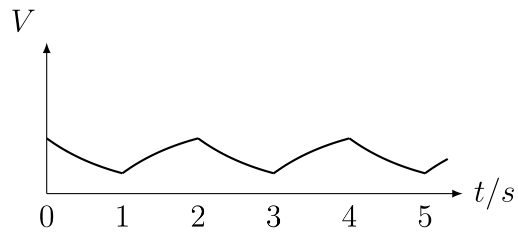
$$t = -RC \ln\left(1 - \frac{\%}{100\%}\right) \tag{4}$$

This gives (i) 42 s (ii) 83 s (iii) 140 s (iv) 690 s.

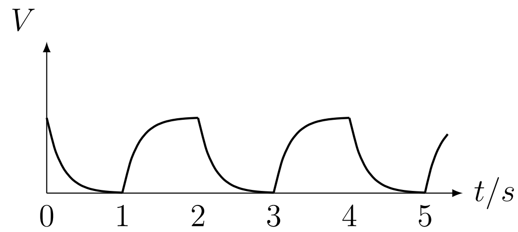
4. (from Nelkon & Parker) The circuit below shows a capacitor C and a resistor R in series. The applied voltage varies with time as shown. The product CR is of the order 1 s. Sketch graphs showing the way the voltages across R and C vary with time. If the product CR were made considerably smaller than 1 s what would be the effect on the graphs?







With $RC = 0.2\,\mathrm{s}$, the graph would be less smooth (as RC gets bigger, the graph tends to a flat line at $V/2$; as $RC \to 0$, the graph tends to its original shape):



5. A 4.5 V battery pack is connected to a 220 µF capacitor through a resistance of 220 kΩ by a student wishing to investigate the charging of a capacitor.

   (a) What is the initial current?

$$I = \frac{V}{R} \tag{5}$$
$$= \frac{4.5\,\mathrm{V}}{220\,\mathrm{k\Omega}} \tag{6}$$
$$= 2.0 \times 10^{-5}\,\mathrm{A}. \tag{7}$$

   (b) What is the current after one minute?

$$I = I_0 e^{-t/RC} \tag{8}$$
$$= 2.0 \times 10^{-5}\,\mathrm{A} \times e^{-60\,\mathrm{s}/(220\,\mathrm{k\Omega} \times 220\,\mu\mathrm{F})} \tag{9}$$
$$= 5.8\,\mu\mathrm{A}. \tag{10}$$

(c) What is the voltage after two minutes?

$$\begin{aligned} V &= V_0 \left(1 - e^{-t/RC}\right) && (11)\\ &= 4.5\,\mathrm{V} \times \left(1 - e^{-120\,\mathrm{s}/(220\,\mathrm{k\Omega} \times 220\,\mathrm{\mu F})}\right) && (12)\\ &= 4.1\,\mathrm{V}. && (13) \end{aligned}$$

(d) How long does it take for the voltage to get to (i) 3.0 V (ii) 4.0 V?

$$\begin{aligned} V &= V_0 \left(1 - e^{-t/RC}\right) && (14)\\ 1 - \frac{V}{V_0} &= e^{-t/RC} && (15)\\ -\frac{t}{RC} &= \ln\left(1 - \frac{V}{V_0}\right) && (16)\\ t &= -RC \ln\left(1 - \frac{V}{V_0}\right) && (17) \end{aligned}$$

So for (i):

$$\begin{aligned} t &= -220\,\mathrm{k\Omega} \times 220\,\mathrm{\mu F} \times \ln\left(1 - \frac{3.0\,\mathrm{V}}{4.5\,\mathrm{V}}\right) && (18)\\ &= 53\,\mathrm{s}. && (19) \end{aligned}$$

and for (ii):

$$\begin{aligned} t &= -220\,\mathrm{k\Omega} \times 220\,\mathrm{\mu F} \times \ln\left(1 - \frac{4.0\,\mathrm{V}}{4.5\,\mathrm{V}}\right) && (20)\\ &= 110\,\mathrm{s}. && (21) \end{aligned}$$

(e) What is the current at each of these times?
Here, we can take a shortcut. Rather than substituting the times just calculated into $I = I_0 e^{-t/RC}$, we can substitute $e^{-t/RC} = \frac{I}{I_0}$ into $V = V_0\left(1 - e^{-t/RC}\right)$ to give

$$\frac{V}{V_0} = 1 - \frac{I}{I_0}. \qquad (22)$$

Hence

$$\begin{aligned} I &= I_0 \left(1 - \frac{V}{V_0}\right) && (23)\\ &= \frac{V_0}{R}\left(1 - \frac{V}{V_0}\right) && (24)\\ &= \frac{V_0}{R} - \frac{V}{R} && (25)\\ &= \frac{V_0 - V}{R} && (26) \end{aligned}$$

So (i) at 53 s when the voltage across the capacitor is 3.0 V,

$$I = \frac{V_0 - V}{R} \tag{27}$$
$$= \frac{4.5\,\text{V} - 3.0\,\text{V}}{220\,\text{k}\Omega} \tag{28}$$
$$= 6.8\,\mu\text{A}, \tag{29}$$

and (ii) at 110 s when the voltage across the capacitor is 4.0 V,

$$I = \frac{4.5\,\text{V} - 4.0\,\text{V}}{220\,\text{k}\Omega} \tag{30}$$
$$= 2.3\,\mu\text{A} \tag{31}$$

(f) How much energy is stored in the capacitor at each of these times?
$E = \frac{1}{2}QV = \frac{1}{2}CV^2 = \frac{1}{2}\frac{Q^2}{C}$. The best version to use in this case is $E = \frac{1}{2}CV^2$, which for (i) gives

$$E = \frac{1}{2}CV^2 \tag{32}$$
$$= \frac{1}{2} \times 220\,\mu\text{F} \times (3.0\,\text{V})^2 \tag{33}$$
$$= 9.9 \times 10^{-4}\,\text{J}, \tag{34}$$

and for (ii) gives

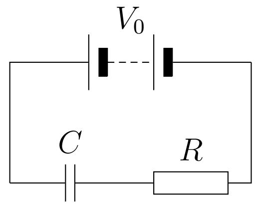
$$E = \frac{1}{2} \times 220\,\mu\text{F} \times (4.0\,\text{V})^2 \tag{35}$$
$$= 18 \times 10^{-4}\,\text{J}. \tag{36}$$

6. Copy and complete the following table. If the graph is not going to be a straight line, write ‘not straight’ instead of the gradient (*Hint: you might like to try Isaac Physics A6*)

Equation	Plotted on y	Plotted on x	y-intercept	Gradient
$Q = Q_0(1 - e^{-t/RC})$	Q	t		
$V = V_0 e^{-t/RC}$	$\ln(V)$	t		
$Q = Q_0(1 - e^{-t/RC})$	Q	t		
$I = I_0 e^{-t/RC}$	I	$e^{t/RC}$		

Equation	Plotted on y	Plotted on x	y-intercept	Gradient
$Q = Q_0(1 - e^{-t/RC})$	Q	t	0	not straight
$V = V_0 e^{-t/RC}$	$\ln(V)$	t	$\ln(V_0)$	$-\frac{1}{RC}$
$Q = Q_0 e^{-t/RC}$	Q	t	Q_0	not straight
$I = I_0 e^{-t/RC}$	I	$e^{-t/RC}$	0	I_0

7. Show that when a battery is used to charge a capacitor through a resistor, the heat dissipated in the resistor in the circuit is equal to the energy stored in the capacitor.



The energy stored in the capacitor when it has fully charged to the battery voltage V_0 is given by $E = \frac{1}{2}C{V_0}^2$.

The energy dissipated in the resistor is given by

$$E = \int P\,\mathrm{d}t, \text{ and since } P = IV = \frac{V^2}{R}, \tag{37}$$

$$= \int \frac{V^2}{R}\,\mathrm{d}t \tag{38}$$

$$= \frac{1}{R}\int_0^\infty {V_0}^2 e^{-\frac{2t}{RC}}\,\mathrm{d}t \tag{39}$$

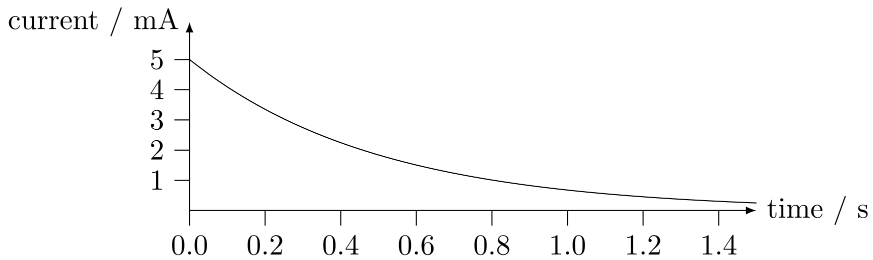
$$= \frac{{V_0}^2}{R}\left[\frac{RC}{2}e^{-\frac{2t}{RC}}\right]_{t=0}^{t=\infty} \tag{40}$$

$$= \frac{{V_0}^2}{R}\frac{RC}{2} \tag{41}$$

$$= \frac{1}{2}C{V_0}^2, \tag{42}$$

which is equal to the energy stored in the capacitor.

8. Examine the graph below, of the current through a capacitor.



(a) Estimate the physical quantity corresponding to the area under the graph.
The area under the graph in this case is the charge stored by the capacitor. Exponentials are so ubiquitous—in electronic circuits, in atmospheric pressure, in radioactive decay—and consequently the area under an exponential occurs so often, it is useful to have a rule-of-thumb (or *heuristic*) to help us integrate it. In dimensionless form, the integral of the exponential e^{-t} is

$$\int_0^\infty e^{-t}\,\mathrm{d}t.$$

To approximate its value, let's lump the e^{-t} curve into one rectangle. What values ought to be chosen for the width and height of the rectangle? A reasonable height for the rectangle is the maximum of e^{-t}, namely 1. To choose its width, use *significant* *change* as the criterion: choose a significant thance in e^{-t}; then find the width Δt that produces this change. In an exponential decay, a simple and natural significant chance is when e^{-t} becomes a factor of e closer to its final value (which is 0 here because t goes to ∞). With this criterion, $\Delta t = 1$. The lumping rectangle then has area of 1 (which is in fact the exact value of the integral!)

Returning to the problem at hand, and using the $1/e$ heuristic outlined above, the charge stored by the capacitor can be estimated by lumping the area under the graph into a rectangle of height 5 mA, and width given by the time taken to fall to $1/e$ of this value (NB this will be one time constant, or RC.) From the graph, the time taken for the current to fall to 1.8 mA is about 0.5 s, and thus the charge stored is $Q = 5\,\text{mA} \times 0.5\,\text{s} = 2.5\,\text{mC}$. Dimensionally, this is indeed a charge, and easy cases also check out, giving us confidence in our answer.

(b) What was the initial current? Was the capacitor charging or discharging?
Although it is easy to read the initial current of 5 mA off the graph, it is impossible to tell whether the capacitor is charging of discharging from the information given.

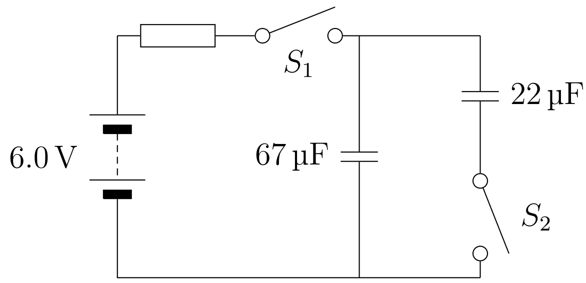
(c) What is the time constant? If the resistor was 470 Ω, what was the capacitance?
The time constant is the time taken for the the current to fall to $5.0\,\text{mA}/e = 1.8\,\text{mA}$, which is about 0.5 s. If the resistor was 470 Ω, since the time constant is given by RC, the capacitance is

$$\frac{0.5\,\text{s}}{470\,\Omega} = 1\,\text{mF}.$$

(d) What voltage is this capacitor charged up to?
The voltage V_0—either the initial voltage across the capacitor to which is has been charged, or the supply voltage to which it is connected—is given by $V_0 = I_0 R = 5\,\text{mA} \times 470\,\Omega = 2.4\,\text{V}$.

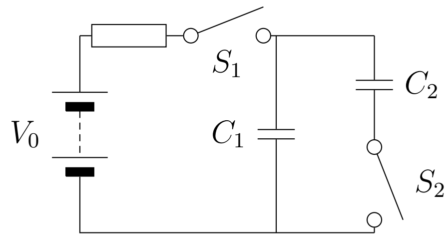
Extension problems

9. (from CEA advanced extension award question) A feature of circuits containing charged capacitors is that there is often a spark at the contacts when a switch is opened or closed. In the circuit below the capacitors are initially uncharged and both switches are open.



Initially, switch S_1 is closed, charging the 67 µF capacitor from the battery. S_1 is then opened, leaving the 67 µF capacitor fully charged and the 22 µF capacitor still uncharged. When switch S_2 is closed, a spark occurs at the contacts of this switch. Estimate the energy dissipated in this spark. Is your value likely to be an over-estimate or an under-estimate? Give a reason.

Let's label the battery voltage as V_0, and relabel the 67 µF capacitor as C_1, and the 22 µF capacitor as C_2:



When S_1 is closed, the capacitor C_1 charges, to a charge $Q = C_1 V_0$. The energy stored in C_1 at this point is given by $E = \frac{1}{2}CV^2 = \frac{1}{2}C_1 {V_0}^2$.

Then S_1 is opened, and S_2 is closed. We want to know the final energy, after some charge has flowed from the plates of C_1 onto those of C_2. What can we say about this final situation? Firstly, the overall charge of the combination must be the same as before the switch S_2 is closed, i.e. $Q_1 + Q_2 = Q$. There must be also be the same voltage across each of the capacitors C_1 and C_2 (let's call it V).

Now let's use these facts. Starting with $Q_1 + Q_2 = Q$, we can see that since $Q_1 = C_1 V$ and $Q_2 = C_2 V$,

$$C_1 V + C_2 V = Q \tag{43}$$

$$(C_1 + C_2)\, V = C_1 V_0 \tag{44}$$

$$V = \frac{C_1 V_0}{C_1 + C_2}. \tag{45}$$

The total energy at the end is given by $E = \frac{1}{2}CV^2$ as

$$E_{\text{final}} = \frac{1}{2}\,(C_1 + C_2)\, V^2 \tag{46}$$

$$= \frac{1}{2}\,(C_1 + C_2)\left(\frac{C_1 V_0}{C_1 + C_2}\right)^2 \tag{47}$$

$$= \frac{1}{2}\frac{{C_1}^2 {V_0}^2}{C_1 + C_2}. \tag{48}$$

The energy loss when switch S_2 is closed gives us an estimate of the energy dissipated in the spark at S_2:

$$E_{\text{spark}} = E - E_{\text{final}} \tag{49}$$

$$= \frac{1}{2}C_1 {V_0}^2 - \frac{1}{2}\frac{{C_1}^2 {V_0}^2}{C_1 + C_2} \tag{50}$$

$$= \frac{1}{2}C_1 {V_0}^2 \left(1 - \frac{C_1}{C_1 + C_2}\right) \tag{51}$$

$$= \frac{1}{2}C_1 {V_0}^2 \left(\frac{C_2}{C_1 + C_2}\right) \tag{52}$$

$$= \frac{1}{2}\frac{C_1 C_2}{C_1 + C_2}{V_0}^2 \tag{53}$$

$$= \frac{1}{2} \times \frac{67\,\mu\text{F} \times 22\,\mu\text{F}}{67\,\mu\text{F} + 22\,\mu\text{F}} \times (6.0\,\text{V})^2 \tag{54}$$

$$= 3.0 \times 10^{-4}\,\text{J}. \tag{55}$$

This is likely to be an overestimate, as *all* of the energy loss will not be in the spark itself; some energy loss will also occur in heating in the wires of the circuit.