On capacitance

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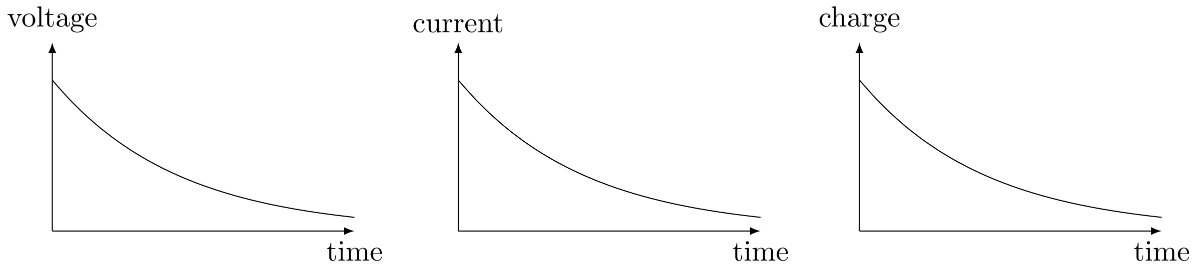
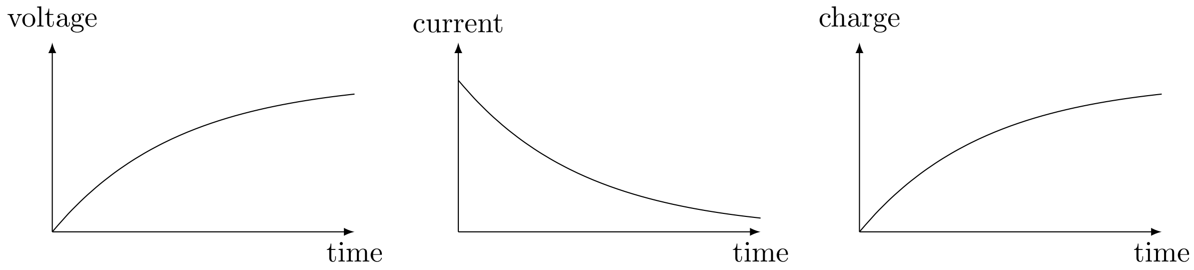
Warm-up problems

1. Sketch graphs of the voltage, current and charge on a capacitor which is being discharged through a resistor.



2. Repeat question 1 for a capacitor in series with a resistor which is being charged by being connected to a battery.



3. What is meant by the *time constant* of a resistor-capacitor combination in charging / discharging? State the expression used to measure it, giving units.

   The time constant of a resistor is a measure of how long you have to wait for most of the charge to run out. Just like radioactivity, we cannot define how long until *all* the charge runs out, as this (in theory) would take forever. A measure of the time taken is given by the product of the resistance of the resistor R and the capacitance of the capacitor C. This combination RC is called the *time constant* of the circuit, and it gives a characteristic time which allows circuits to be compared (in fact, it is the time for the charge to fall to $\frac{1}{e}$ of the original value). The time constant has units of resistance multiplied by capacitance ($\Omega\,\mathrm{F} = \Omega\frac{\mathrm{C}}{\mathrm{V}} = \frac{\mathrm{V}}{\mathrm{A}}\frac{\mathrm{C}}{\mathrm{V}} = \frac{\mathrm{C}}{\mathrm{A}} = \frac{\mathrm{C}}{\mathrm{C/s}} = \mathrm{s}$)

Regular problems

4. Copy and complete the time constant table shown below

C	R	time constant
1 µF	1 MΩ	
	1 kΩ	1 s
0.22 µF	150 kΩ	
10 µF		0.47 s

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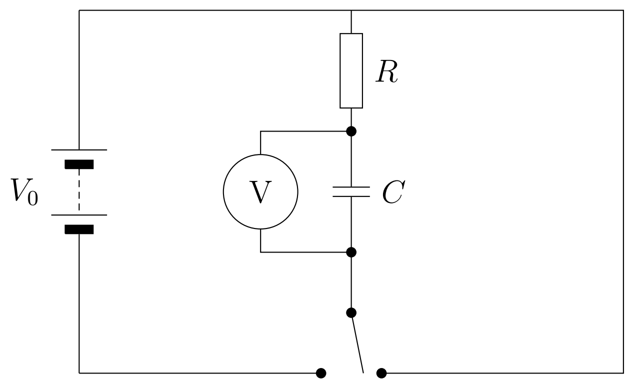
1 µF	1 MΩ	1 s
1 mF	1 kΩ	1 s
0.22 µF	150 kΩ	33 ms
10 µF	47 kΩ	0.47 s

5. Design a circuit which could be used to investigate the charging and discharging of a capacitor, with appropriate meters for measurement of the relevant quantities.



6. Calculate the energy stored in

   (a) a capacitor with a charge of 2 C and 9 V across its plates,

$$
\begin{aligned}
E &= \frac{1}{2}QV && (1)\\
&= \frac{1}{2} \times 2\,\mathrm{C} \times 9\,\mathrm{V} && (2)\\
&= 9\,\mathrm{J} && (3)
\end{aligned}
$$

   (b) a 1 µF capacitor charged to 500 V.

$$
\begin{aligned}
E &= \frac{1}{2}CV^2 && (4)\\
&= \frac{1}{2} \times 1\,\mu\mathrm{F} \times (500\,\mathrm{V})^2 && (5)\\
&= 0.13\,\mathrm{J}\text{, or even just } 0.1\,\mathrm{J} && (6)
\end{aligned}
$$

7. A 1.5 V battery is connected to a 1000 µF capacitor in series with a 150 Ω resistor.

   (a) What is the maximum current which flows through the resistor during charging?

$$
\begin{aligned}
I &= \frac{V}{R} && (7)\\
&= \frac{1.5\,\mathrm{V}}{150\,\Omega} && (8)\\
&= 0.01\,\mathrm{A} && (9)
\end{aligned}
$$

   (b) What is the maximum charge on the capacitor?

$$
\begin{aligned}
Q &= CV && (10)\\
&= 1000\,\mu\mathrm{F} \times 1.5\,\mathrm{V} && (11)\\
&= 1.5\,\mathrm{mC} && (12)
\end{aligned}
$$

   (c) How long does the capacitor take to charge to 1.0 V?

$$
\begin{aligned}
V &= V_0\left(1 - e^{-t/RC}\right) && (13)\\
\frac{V}{V_0} - 1 &= -e^{-t/RC} && (14)\\
1 - \frac{V}{V_0} - 1 &= e^{-t/RC} && (15)\\
\ln\left(1 - \frac{V}{V_0}\right) &= -\frac{t}{RC} && (16)\\
t &= -RC\ln\left(1 - \frac{V}{V_0}\right) && (17)\\
&= -150\,\Omega \times 1000\,\mu\mathrm{F}\ln\left(1 - \frac{1.0\,\mathrm{V}}{1.5\,\mathrm{V}}\right) && (18)\\
&= 0.16\,\mathrm{s} && (19)
\end{aligned}
$$

8. A 5 µF capacitor is charged up to a p.d. of 10 V. It is then removed from the charging circuit and connected across an uncharged 2 µF capacitor. Calculate

   (a) the initial charge and energy stored in the 5 µF capacitor

$$
\begin{aligned}
Q &= CV && (20)\\
&= 5\,\mu\mathrm{F} \times 10\,\mathrm{V} && (21)\\
&= 50\,\mu\mathrm{C}. && (22)
\end{aligned}
$$

$$
\begin{aligned}
E &= \frac{1}{2}CV^2 && (23)\\
&= \frac{1}{2} \times 5\,\mu\mathrm{F} \times (10\,\mathrm{V})^2 && (24)\\
&= 250\,\mu\mathrm{J}. && (25)
\end{aligned}
$$

(b) the final p.d. across the combination
The key here is that the charge on the 'new' combined capacitor is the same. The capacitance of the new combination is given (for parallel capacitors) by $C_{\text{total}} = C_1 + C_2$,[1] so in this case $5\,\mu\text{F} + 2\,\mu\text{F} = 7\,\mu\text{F}$.
The final p.d. can be obtained from this by using

$$Q = CV \tag{26}$$
$$V = \frac{Q}{C} \tag{27}$$
$$= \frac{50\,\mu\text{C}}{7\,\mu\text{F}} \tag{28}$$
$$= 7.1\,\text{V}. \tag{29}$$

(c) the energy stored by the capacitors at the end. Why is this less than you started with?

$$E = \frac{1}{2}\frac{Q^2}{C} \tag{30}$$
$$= \frac{1}{2} \times \frac{(50\,\mu\text{C})^2}{7\,\mu\text{F}} \tag{31}$$
$$= 180\,\mu\text{J}. \tag{32}$$

This is less than we started with as it took energy to move the charge from one capacitor to the other. $70\,\mu\text{J}$ of energy has been dissipated in the wires connecting the two capacitors.

9. A student measures the capacitance of a capacitor by placing it in a circuit in series with a $1\,\text{M}\Omega$ resistor and a $48\,\text{V}$ battery. He observes, that $5\,\text{s}$ after closing the switch, the voltage across the capacitor is $33\,\text{V}$. What is the capacitance?

$$V = V_0\left(1 - e^{-\frac{t}{RC}}\right) \tag{33}$$
$$\ln\left(1 - \frac{V}{V_0}\right) = -\frac{t}{RC} \tag{34}$$
$$C = -\frac{t}{R}\frac{1}{\ln\left(1 - \frac{V}{V_0}\right)} \tag{35}$$
$$C = -\frac{5\,\text{s}}{1\,\text{M}\Omega}\frac{1}{\ln\left(1 - \frac{33\,\text{V}}{48\,\text{V}}\right)} \tag{36}$$
$$= 4.3\,\mu\text{F}. \tag{37}$$

[1] You could derive this by imagining a certain voltage across the combination, and figuring out what charge would be stored.

Extension problems

10. A voltmeter has a range of 0 to 250 V and the smallest potential difference that can be estimated with is is 0.5 V. A 1.0 µF capacitor is charged to 200 V and then allowed to discharge through a 1.0 MΩ resistor. In what time will the capacitor be completely discharged as indicated by the voltmeter?

$$V = V_0 e^{-\frac{t}{RC}} \tag{38}$$
$$t = RC\,(\ln V_0 - \ln V) \tag{39}$$
$$= 1.0\,\mathrm{M\Omega} \times 1.0\,\mathrm{\mu F} \times (\ln 200\,\mathrm{V} - \ln 0.5\,\mathrm{V}) \tag{40}$$
$$= 6.0\,\mathrm{s}. \tag{41}$$

11. Two metal plates, each having area $0.05\,\mathrm{m}^2$ are mounted 2 mm apart in a vacuum. This arrangement is found to have a capacitance of 220 pF. The plates are then given a charge of 0.1 µC.

    (a) What is the potential difference between the plates?

$$V = \frac{Q}{C} \tag{42}$$
$$= \frac{0.1\,\mathrm{\mu C}}{220\,\mathrm{pF}} \tag{43}$$
$$= 450\,\mathrm{V}. \tag{44}$$

    (b) What is the electric field strength between the plates?

$$E = \frac{V}{d} \tag{45}$$
$$= \frac{450\,\mathrm{V}}{2\,\mathrm{mm}} \tag{46}$$
$$= 230\,\mathrm{kV\,m^{-1}}. \tag{47}$$

    (c) How much energy is stored in this system?

$$E = \frac{1}{2}\frac{Q^2}{C} \tag{48}$$
$$= \frac{1}{2} \times \frac{(0.1\,\mathrm{\mu C})^2}{220\,\mathrm{pF}} \tag{49}$$
$$= 23\,\mathrm{\mu J}. \tag{50}$$

    (d) If the separation of the plates is now doubled, what is the effect on the field between the plates, the potential of the plates, the capacitance and the electrical stored energy? How can this change in energy be accounted for?

$$C \propto \frac{1}{d},$$

so the capacitance will decrease by a factor of 2.

Assuming that the plates are isolated before doubling the separation of the plates (so charge Q stays the same as it cannot move on or off the plates),

$$V \propto \frac{1}{C},$$

so the voltage will increase by a factor of 2. Since

$$E = \frac{V}{d},$$

and both V and d have increased by a factor of 2, the field will not change. The volume of the field has doubled though, so the energy stored in it will have doubled too (alternatively, consider that the energy stored in the capacitor $= \frac{1}{2}QV$, and Q is unchanged whilst V has doubled).

Where has this extra energy come from? A force is required to separate the plates of the capacitor, as they have equal and opposite charges $+Q$ and $-Q$, and therefore work has been done in moving the plates apart. This creates the extra field energy.