Train Paradox

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Abstract

An account is given of the apparent paradoxical symmetry of length contraction in special relativity, which on first inspection may appear to be difficult to reconcile with classical ideas about the measurement of length. A thought experiment, to 'prove' the Lorentz-Fitzgerald contraction, is proposed, and carefully analysed in the frames of reference of the observer and of the object whose contracted length is measured. The apparent paradox is resolved, leaving a deeper insight into the loss of simultaneity between frames in special relativity. Finally, the question of whether a physical objects can be trapped inside a box which is shorter than its proper length is considered, and the results reasoned in one frame are shown to be true in all frames.

1 Introduction

According to special relativity, an observer in an inertial frame of reference[1] will measure an object moving relative to his frame to have a shorter length in the direction of its motion than it would have if it were at rest with respect to his inertial frame. This phenomenon is know as the Lorentz-Fitzgerald length contraction.

The symmetry of the situation suggests that a second observer riding along with the object will see the first observer, and all objects at rest with respect to that observer's frame, as length contracted. A paradox immediately springs to mind, that each observer could attempt to confine an object, which is moving in their frame, but stationary in the other frame, in a space shorter than its proper length. This paper will attempt to answer the question, "Is the symmetry of length contraction paradoxical?"

[1] An *inertial frame of reference* is one which is not accelerating

2 Background: length contraction

$$L = \frac{L_0}{\gamma} = \sqrt{1 - \frac{v^2}{c^2}}\, L_0$$

3 The paradox

Suppose that one observer is determined to prove the reality of the Lorentz-Fitzgerald contraction.

4 Resolution

The answer is buried in the misuse of the word "simultaneously" in this story. In special relativity, events separated in space which appear simultaneous in one frame of reference need not appear simultaneous in another frame. The closing doors of the tunnel ends are two such separate events.

Imagine that, as the train passes through the tunnel, when it is completely inside the tunnel, both doors are closed, but just for a moment. At that instant, both doors could be closed simultaneously, with a switch, and at least momentarily the contracted train would be shut up inside the tunnel, proving that it has been contracted.

Special relativity explains that the two doors would not be closed at the same time in the train's frame of reference, so there is always room for the train. In fact, the Lorentz transformation for time is

$$t' = \frac{t - vx}{c^2} \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

.

It's the vx term in the numerator that causes the mischief here. In the train's frame the further event (larger x) happens earlier–the far door is closed first. It opens before the train hits it, the train moves through the door, and then the other door closes behind it.

Second paradox Suppose that a train robber decides to stop a train inside tunnel. The proper length of the train is 60 m, and the proper length of the tunnel is 50 m. The train is traveling at 4/5 the speed of light. If both the train and tunnel were at rest, the train could not fit inside the tunnel, as we know from the proper lengths of both objects. But in the rest frame of the tunnel, and of the robber, the length of the moving train is 36 m. The

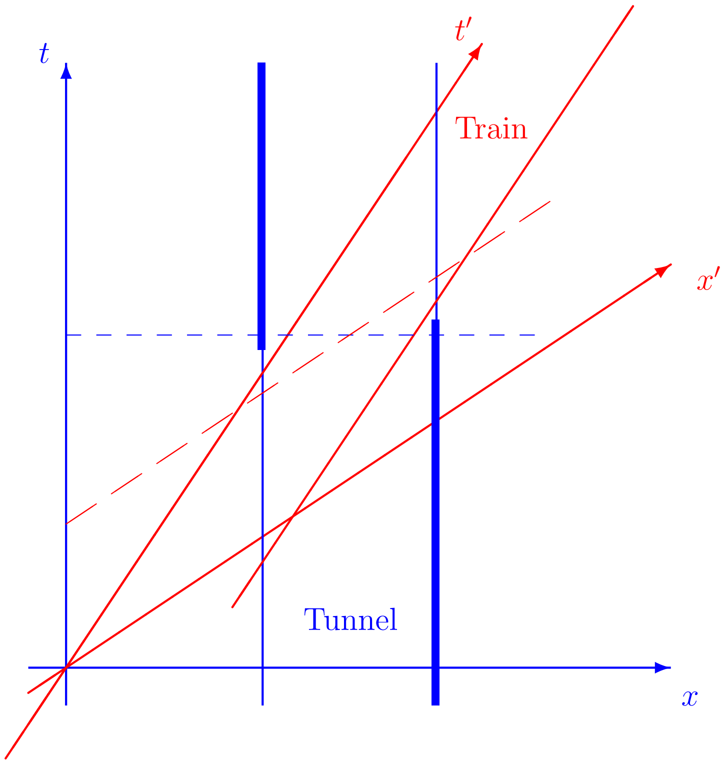

Figure 1: A space-time diagram of the situation. In this diagram, the stationary frame is that of the tunnel, the moving frame is the train, and thick lines represent the closed doors of the tunnel. Dashed lines show lines of constant time, and it can be seen that in the tunnel's frame, there is a time when both doors are closed with the train trapped inside. In the train's frame, this never happens–one door opens before the other closes–and there is a time when both ends of the train stick out of the tunnel; in this frame, the tunnel's length, projected onto the x' axis, appears shorter than the train's.

robber computes this, and decides to trap the train inside the tunnel, using special relativity, as the train ought to fit inside the tunnel in his frame.

From the point of view of a passenger on the train, however, the tunnel is length contracted, and appears to be only 30 m long–just half the length of the train, which of course has its proper length in this frame. Therefore, such an observer would conclude that the 60 m long train will not fit completely into the 30 m long tunnel. Of course, the passenger and the robber cannot both be correct, since the train will either become trapped or it will not.

We do not doubt the feasibility if this experminent, and in the tunnel's frame, we expect the train to become trapped in the tunnel. A massive door indeed will be required to bring the train to a halt, but let us presume that the train will come to a standstill in the inertial frame of the tunnel. When the train stops in the rest-frame of the tunnel, it will tend to assume its original length relative to the tunnel. If it has survived the deccelleration, it must bend, burst the door, or remain compressed.

Let us now consider things from the train's point of view. It is true that the train, of proper length 60 m, will measure the tunnel as having a length of 30 m. The 30 m long tunnel will rush towards the stationary train. Because of the closed door, it will continue rushing onward even after the train's collision, taking the front end of the train with it. The back end of the train, however, is still at rest: it cannot 'know' that the front end has been struck, because of the finite speed of propogation of all signals. Even if the 'signal' (in this case an elastic shockwave) travels at the speed of light, the signal has 60 m to travel, twice as far as the tunnel's other end at the moment of impact in this frame, before reaching the other end of the train. The signal and the tunnel's other end would arrive at the same time if v were $0.5c$, but v is $\frac{4}{5}c$, so the train *more* than just gets in.

> There is one important moral to this story: whatever result we get by correct reasoning in one inertial frame, must be true; in particular, it must be true when viewed from any other inertial frame. As long as the set of physical laws we are using is self-consistent and Lorent-invariant, there *must* be an explanation of the result in every other inertial frame, although it may be quite a different explanation from that in the first frame.
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> —Rindler, *Relativity*

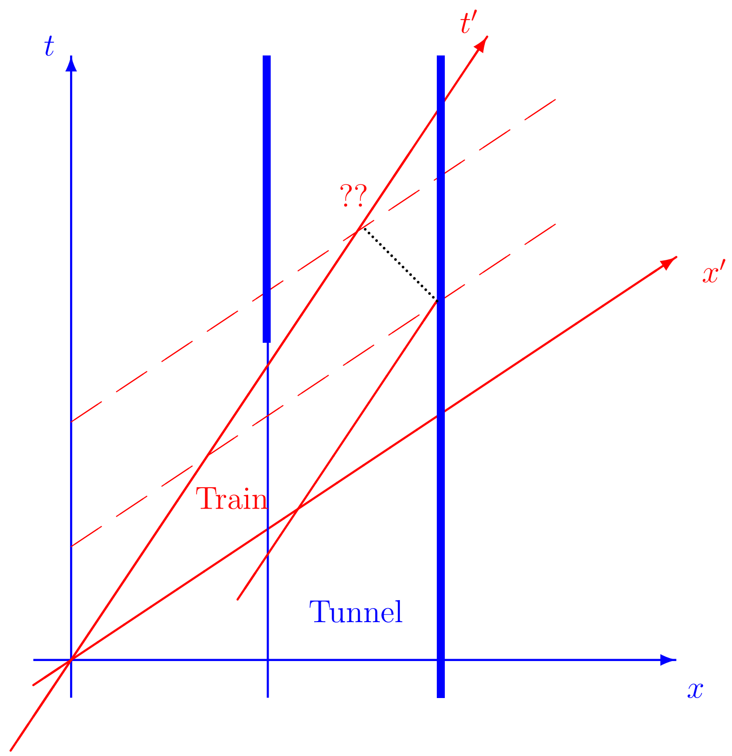

Figure 2: A space-time diagram to show the tunnel trapping the contracted train. Once again, the stationary frame is that of the tunnel, the moving frame is the train, and thick lines represent the closed doors of the tunnel. Dashed lines show lines of constant time, and it can be seen at the instant that the tain hits the end of the tunnel, the other end of the train has not yet entered the tunnel in the train's frame, whereas in the tunnel's frame, the other door has already closed. The dotted line shows the fastest possible transmission of information regarding the train's collision, and we can see that when this reaches the other end of the train, that end is well inside the tunnel, and the other door is closed, in both frames. What happens next depends on the material properties of the train.