CONSERVATION METHODS IN DYNAMICS

Introduction

The usual path to solving the motion in time of a particle, $x(t)$, is via the forces acting on it. One employs $F = ma$ (Newton's second law). Recognising that $a = \frac{d^2x}{dt^2}$, rearrangement then gives a second order differential equation $\frac{d^2x}{dt^2} = \frac{F(x)}{m}$, where the force is a function of position. This is a rather formal statement of a familiar law that is often simple to apply. Formally it tells us that two integrations must be carried out to obtain $x(t)$ from it. Force derives from the spatial variation of potential energy, $F = -\frac{dV}{dx}$. Under these circumstances it is always possible to simplify to obtain one of the two integrals along the path from $F = ma$ to $x(t)$. It is a restatement of the conservation of energy- see Exercise 6 below. If the total energy E is conserved, then $E = constant = T + V$, where T is the kinetic energy, $T = \frac{1}{2}mv^2$ where v is the speed, $v = \frac{dx}{dt}$. Thus one can solve for the speed: $v(x) = \sqrt{\frac{2(E-V(x))}{m}}$. Further $dt = \frac{dx}{v(x)}$ and thus the relation between x and t is reduced to:

$$t = \int_0^t dt' = \int_{x_0}^{x} \frac{dx'}{v(x')} = \int_{x_0}^{x} \sqrt{\frac{m}{2(E - V(x'))}} dx'$$

where the particle is at x_0 at $t = 0$ and at x at time t. One last integral needs to be performed to obtain $x(t)$. Below, we practise cases where the integral is easy, usually by a substitution. In general this last step is not possible analytically and numerical computation ensues.

It is assumed that students will be familiar with the following concepts:

- Conservation of potential plus kinetic energy.
- Solving for speed as a function of position using energy conservation.
- Calculating time elapsed during motion to a spatial point by integrating the differential relation $\mathrm{d}t = \mathrm{d}x/v(x)$.
- Integration by substitution.

Questions

Exercise 1: Consider a particle of mass m passing a potential well of width a, as shown in Fig. 1. The particle has total energy $E > V_0$, the depth of the well. Calculate the time taken by the particle to traverse the figure. [CQMP]

Exercise 2: A particle of mass m slides down, under gravity, a smooth ramp which is inclined at angle θ to the horizontal. At the bottom, it is joined smoothly to a similar ramp rising at the same angle θ to the horizontal to form a V-shaped surface. If the particle slides smoothly around the join, determine the period of oscillation, T, in terms of the initial horizontal displacement x_0 from the centre join. Note the shape of the potential well.

Hint: We see that the potential well appears as a sloping line similar to the one along which the particle is

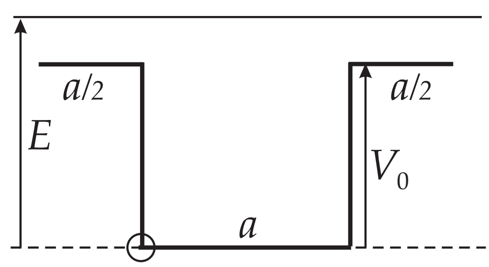

Figure 1: A finite square well potential of depth V_0.

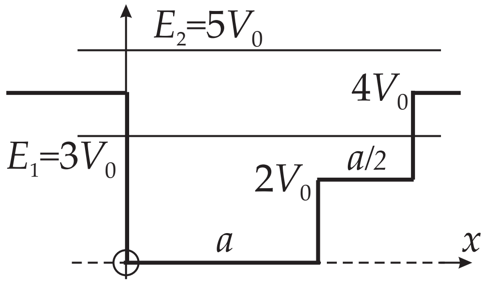

Figure 2: A stepped rectangular potential well

constrained to move. It is only this linear slope at angle θ to the horizontal, that happens to resemble the potential energy graph of the same shape, which misleads us into thinking that we can see the potential energy. The potential energy is a concept, represented pictorially by a graph and the shape of the graph happens, in some cases, to resemble the mechanical system. [CQMP]

Exercise 3: A particle moves in a potential $V(x) = \frac{1}{2}qx^2$. If it has total energy $E = E_0$ give an expression for its velocity as a function of position $v(x)$. What is the amplitude of its motion? [CQMP]

Exercise 4: The potential energy of a particle of mass m as a function of its position along the x axis is as shown in Fig. 2.
(a) Sketch a graph of the force versus position in the x direction which acts on a particle moving in this potential well with its vertical steps. Why is this potential unphysical?
(b) Sketch a more realistic force versus position curve for a particle in this potential well. For a particle moving from $x = 0$ to $x = \frac{3a}{2}$, which way does the force act on the particle? If the particle was moving in the opposite direction, which way would the force be acting on the particle?
(c) Calculate the period for a complete oscillation of the particle if it has a total mechanical energy E equal to $3V_0$.
Hint: Take care over the physical meaning of the potential energy. It can look misleadingly like the physical picture of a particle sliding off a high shelf, down a very steep slope and then sliding along the floor, reflecting off the left hand wall and then back up the slope. This is too literal an interpretation since, for example, the potential change might be due to an electrostatic effect rather than a gravitational one, and the time spent moving up or down the slope is due to artificially putting in an extra vertical dimension in a problem which is simply about motion in only one dimension. An example of where there is literally motion vertically as well as horizontally, is that of a frictionless bead threaded on a parabolic wire. The motion is not the same as in the one-dimensional simple harmonic motion of Ex. 3. Although the potential energy is expressible in the form $\frac{1}{2}qx^2$ due to the constraint of the wire, the kinetic energy involves both the x and y variables. [CQMP]

Exercise 5: A particle of energy $E_2 = 5V_0$ approaches the potential of Fig. 2. How long does it take to travel from $-a$ to $+2a$? [CQMP]

Exercise 6: A particle with energy E incident from $x < 0$ on a potential ramp of the form $V(x) = 0$ for $x < 0$ and $V(x) = V_0 x/d$ for $x \in (0, d)$ and $V = V_0$ for $x > d$. For $E < V_0$ give the x value of the classical turning point, x_{ctp}, where the particle briefly stops before turning back. Show that the time taken to travel from $x = 0$ to a point x_0 on the ramp ($x_0 \leq x_{\mathrm{ctp}}$) is $t_0 \propto (1 - \cos\theta_0)$, where $\cos\theta_0 = (1 - \frac{V_0 x_0}{Ed})^{1/2}$. Give the constant of proportionality. What is the time taken to reach the classical turning point?
Hint: The form of the answer is a steer to the integration by substitution that is involved.
Further insight comes from considering the form of the force implied by this potential. Solve this elementary problem instead by integration of Newton's Second Law (that is, a method involving forces) and show that the answer is the same as above. [CQMP]

Exercise 7: Consider a particle with energy E incident from $x < 0$ on a potential $V(x) = 0$ for $x < 0$ and $V(x) = \frac{1}{2}V_0(x/d)^2$ for $x > 0$. Again calculate x_{ctp}. Show that the time taken to reach a position

$0 < x_0 \leq x_{\rm ctp}$ from $x = 0$ is $t_0 \propto \sin^{-1}\left(\sqrt{\frac{V_0}{2E}}\frac{x_0}{d}\right)$, and give the constant of proportionality. What is important about the E-dependence of the time taken to reach the classical turning point? Evaluate and interpret this time. [CQMP]

Exercise 8: A particle of mass m is constrained to slide along a smooth wire lying along the x axis, as shown in Figure 3. The particle is attached to a spring of natural length l_0 and spring constant q which has its other

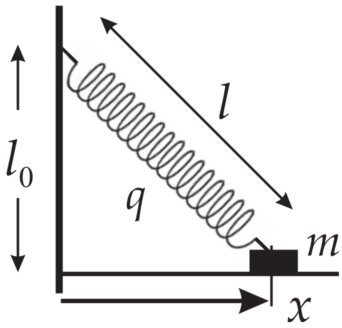

Figure 3: A constrained mass on a spring

end fixed at $x = 0$, $y = l_0$.
(a) Obtain an expression for the force exerted on m in the x direction.
(b) For small displacements ($x \ll l_0$), how does the force depend upon displacement x?
(c) The potential $U(x)$ depends upon x in the form of $U \simeq Ax^n$ for small x. What are the values of n and A in terms of the constants given?
(d) Find the exact potential.
(e) By sketching a graph of the potential energy, suggest qualitatively how the period of oscillation of the object will depend on the amplitude.
(f) For $n = 4$ and amplitude x_0, show that the period is

$$\tau = 4\frac{1}{x_0}\sqrt{\frac{m}{2A}}\int_0^1 \frac{\mathrm{d}u}{\sqrt{1-u^4}}.$$

[CQMP]

Exercise 9: Consider a particle with energy E incident from $x < 0$ on a potential $V(x) = 0$ for $x < 0$ and $V(x) = -\frac{1}{2}V_0(x/d)^2$ for $x > 0$. Show the time taken to reach a position $x_0 > 0$ from $x = 0$ is $t_0 \propto \sinh^{-1}\left(\sqrt{\frac{V_0}{2E}}\frac{x_0}{d}\right)$. Give the constant of proportionality. Examine the small time variation of position and how x_0 increases for large times. [CQMP]

Exercise 10: Consider a particle of energy E in a potential with a special, matching amplitude $V(x) = E\sin^2(\frac{x}{\sigma})$. If the particle starts at $x = 0$, show that at time t it has travelled a distance x given by $\tan\left(\frac{x}{2\sigma}\right) = \tanh\left(\sqrt{\frac{2E}{m}}\frac{t}{2\sigma}\right)$. Be sure to plot $V(x)$ and show on the same graph the particle energy. Discuss the apparent difficulty when $\frac{x}{2\sigma} \to \frac{\pi}{2}$. [CQMP]

Exercise 11: Consider the special case of motion of a particle of energy E in a potential $V(x) = -V_0\sinh^2(\frac{x}{\sigma})$ but when $V_0 = E$. (Note that σ sets the spatial scale of $V(x)$.) Given that the particle is initially at $x = 0$, find how far it travels as a function of the time t. Comment on the time at which the particle is at infinity. [CQMP]

[MW; Oct. 13]