Practical Physics

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1 Errors, and how to make them

Every dog has its day, every silver lining has its cloud, and every measurement has its error.[1]

If you doubt this, take (sorry – borrow with permission) a school metre stick, and try and measure the length of a corridor in your school. Try and measure it to the nearest centimetre. Then measure it again. Unless you cheated by choosing a short corridor, you should find that the measurements are different. What's gone wrong?

Nothing has gone wrong. No measurement is exact, and if you take a series of readings, you will find that they cluster around the 'true value'. This spread of readings is called random error and will be determined by the instrument you use and the observation technique. To be more precise and polite, this kind of 'error' is usually called uncertainty, as this word doesn't imply any mistake or incompetence on the part of the scientist.

So, whenever you write down a measurement, you should also write down its uncertainty. This can be expressed in two ways – absolute and relative.

The absolute uncertainty gives the size of the spread of readings. You might conclude that your corridor was (12.3 ± 0.2) m long. In other words, your measurements are usually within 20 cm of 12.3 m. In this case the absolute uncertainty is 20 cm.

The absolute uncertainty only gives part of the story. A 10 cm error in the length of a curtain track implies sloppy work. A 10 cm error in the total length of the M1 motorway is an impressive measurement. To make this clearer, we often state errors (or uncertainties) in percentage form – and this is called relative uncertainty. The relative uncertainty in the length of the corridor is

$$\text{Relative Error} = \frac{\text{Abs. Uncertainty}}{\text{Measurement}} = \frac{0.2\text{ m}}{12.3\text{ m}} = 1.6\% \approx 2\%$$

Notice the rounding off at the end. It is usually pointless to give uncertainties to more than one significant figure.

[1] A mathematician would probably be appalled at some of the statements I make. The study of errors and uncertainties is embedded in statistics, which is a well-established discipline. There are many refinements to the results I quote which are needed to satisfy the rigour of a professional statistician. However, the thing about uncertainties in measurements is that quoting them to more than one significant figure is missing the point, and therefore our methods only need to be accurate to this degree. If you are doing statistics and you want to take things more seriously, then you will understand (2) from the addition of variances; and you will realise that in 8.2.1 we really ought to be adding variances not errors. You will also appreciate that (2) ought to have an $(n-1)$ in the denominator to take into account the difference between population and sample statistics, and that our section 8.2.2 is a form of the Binomial theorem to first order.

Every measurement has its uncertainty, and the only way of determining this is to take more than one measurement, and work out the standard deviation – to measure the spread. In practice the spread can be 'eyeballed' rather than calculated. If the measurements were 54.5 cm, 54.7 cm and 54.3 cm, then there is no need to use a calculator and the technical definition of deviation. The observation that the spread is about ± 0.2 cm is perfectly good enough.

Notice that the more readings you take, the better idea you get of the spread of the measurements – and hence the better estimate you can make for the middle, which is indicative of 'true' value. Therefore we find, from statistics, that if you take n measurements, and the absolute uncertainty is x, then the uncertainty of the mean of those measurements is approximately:[2]

$$\text{Uncertainty of mean} = \frac{x}{\sqrt{n}}.$$

Therefore, the more measurements you take, the more accurate the work. Notice that if you wish to halve the uncertainty, you need to take *four* times as many readings. This is subject to one proviso:

Measurements also have a resolution. This is the smallest distinguishable difference that the measuring device (including the technique) can detect. For a simple length measurement with a metre ruler, the resolution is probably 1 mm. However if, by years of practice with a magnifying lens, you could divide millimetres into tenths by eye, you would have a resolution of 0.1 mm using the same metre stick. That is why we say that the resolution depends on the technique as well as on the apparatus.

The uncertainty of a measurement can never be less than the resolution. This is the proviso we mentioned. Why should this be the case? Let us have a parable.

> Many years ago, the great nation of China had an emperor. The masses of the population were not permitted to see him. One day, a citizen had the sudden desire to know the length of the emperor's nose. He could not do this directly, since he was not permitted to visit the emperor. So, using the apparatus of the imperial administration, he asked all the regional mandarins to ask the entire population to make a guess. Each person would make some guess at the imperial nasal length – and the error of each guess would probably be no more than ± 2 cm – since nose lengths tend not to vary by more than about 4 cm.
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> However, the mean would be a different matter. Averaged over the 1000 million measurements, the error in the mean would be 0.7 μm. So the emperor's nose had been measured incredibly accurately – without a single observation having been made!

The moral of the story: uncertainties are reduced by repeated measurement, but the error can never be reduced below the resolution of the technique—here 2 cm—since ignorance can not be circumvented by pooling it with more ignorance.

[2] This result will be proved in any statistics textbook. To give a brief justification – the more readings you take, the more likely you are to have some high readings cancelling out some low readings when you take the average.

2 Errors, and how to make them worse

Errors are one thing. The trouble is that usually we want to put our measurements into a formula to calculate something else. For example, we might want to measure the strength of a magnetic field by measuring the force on a current-carrying wire $B = \frac{F}{IL}$.

If there is a 7% uncertainty in the current, 2% in the force and 1% in the length – what is the uncertainty in the magnetic field?

There are two rules you need:

2.1 Rule 1 Adding or subtracting measurements

If two measurements are added or subtracted, the absolute uncertainty in the result equals the sum (never the difference) of the absolute uncertainties of the individual measurements.

Therefore if a car is (3.2±0.1) m long, and a caravan is (5.2±0.2) m long, the total length is (8.4±0.3) m long. Similarly if the height of a two-storey house is (8.3±0.2) m and the height of the ground floor is (3.1±0.1) m, the height of the upper floor is (5.2±0.3) m.

Even in the second case, we do not subtract the uncertainties, since there is nothing stopping one measurements being high, while the other is low.[3]

2.2 Rule 2 Multiplying or dividing measurements

If two measurements are multiplied or divided, the **relative** uncertainty in the result equals the sum (never the difference) of the relative uncertainties of the individual measurements.

Therefore if the speed of a car is 30 mph±10%, and the time for a journey is 6 hours±2%, the uncertainty in the distance travelled is 12%.

Notice that one consequence of this is that if a measurement, with relative uncertainty $p\%$ is squared (multiplied by itself), the relative error in the square is $2p\%$, i.e. doubled. Similarly if the error in measurement L is $p\%$, the error in L^n is $p \times n\%$. Notice that while a square root will halve the relative error, an inverse square ($n = 2$) doubles it. All the minus sign does is to turn overestimates into underestimates. It does not reduce the magnitude of the relative error.[4]

Now we can answer our question about the magnetic field measurement at the beginning of the section. All three relative errors (in length, force and current) must be added to give the relative error in the magnetic field, which is therefore 10%.

[3] Of course, there is a good chance that the errors will partly cancel out, and so our method of estimating the overall error is pessimistic. Nevertheless, this kind of error analysis is good enough for most experiments – after all it is better to overestimate your errors. If you want to do more careful analysis, then you work on the principle that if the absolute uncertainties in a set of measurements are A, B, $C\ldots$, then the absolute uncertainty in the sum (or in any of the differences) is given by $\sqrt{A^2 + B^2 + c^2 \ldots}$. This result comes from statistics, where we find that the variance (the square of the standard deviation) of a sum is equal to the sum of the variances of the two measurements.

[4] The conclusions of this paragraph can be justified using calculus. If measurement x has absolute uncertainty δx, and y (a function of x) is given by $y = Ax^n$, then we find that the relative error in y is given by:

$$\frac{\delta y}{y} \approx \frac{dy}{dx}\frac{\delta x}{y} = \left(Anx^{n-1}\right)\frac{\delta x}{y} = n\frac{y}{x}\frac{\delta x}{y} = n\frac{\delta x}{x}$$

that is n multiplied by the relative error in x.

3 Systematic Errors

All the 'errors' mentioned so far are called 'random', since we assume that the measurements will be clustered around the true value. However often an oversight in our technique will cause a measurement to be overestimated more often than underestimated or vice-versa. This kind of error is called 'systematic error', and can't be reduced by averaging readings. The only way of spotting this kind of error (which is a true error in that there is something wrong with the measurement) is to repeat the measurement using a completely different technique, and compare the results. Just thinking hard about the method can help you spot some systematic errors, but it is still a good idea to perform the experiment a different way if time allows.

4 Which Graph?

You will often have to use graphs to check the functional form of relationships. You may also have to make measurements using the graph. In order to do either of these, you usually need to manipulate the data until you can plot a straight line. A straight line is conclusive proof that you have got the form of the formula right!

The gradient and y-intercept can then be read, and these enable other measurements to be made. For example, your aim may be to measure the acceleration due to gravity. You may plot velocity of falling against time, in which case you will need to find the gradient of the line.

At its most general, you will have a suspected functional form $y = f(x)$, and you will need to work out what is going on in the function f. Notice that our experiment will give us pairs of (x, y) values – what is not known are the parameters in the function f. We find them by manipulating the equation:

$$\begin{aligned} y &= f(x) \\ &\vdots \\ g(x,y) &= Ah(x,y) + B \end{aligned}$$

We can then plot $g(x, y)$ against $h(x, y)$, and obtain the parameters A and B from the gradient and intercept of the line. Furthermore, the presence of the straight line on the graph assures us that our function f was a good guess. We shall now look at the most common examples.

4.1 Exponential growth or decay

Here we have the functional form $y = Ae^{Bx}$, where A and B need to be determined. We manipulate the equation:

$$\begin{aligned} y &= Ae^{Bx} \\ \ln y &= \ln A + Bx \end{aligned}$$

So we plot $(\ln y)$ on the vertical axis, and x on the horizontal. The y-intercept gives $\ln A$, and the gradient gives B.

4.2 Logarithmic growth or decay

Here we have the functional form $y = A + B \ln x$, and again we need to work out the values of A and B. This equation is already in linear form – we plot y on the vertical, and $(\ln x)$ on the horizontal. The y-intercept gives A, and the gradient gives B.

4.3 Power laws

This covers all equations with unknown powers: manipulation involves logarithms:

$$\begin{aligned} y &= Ax^B \\ \ln y &= \ln A + \ln x^B \\ \ln y &= \ln A + B \ln x \end{aligned}$$

Here we plot $(\ln y)$ against $(\ln x)$, and find the power B as the gradient of the line. The A value can be inferred from the y-intercept, which is equal to $\ln A$.

4.4 Other forms

Even hideous looking equations can be reduced to straight lines if you crack the whip hard enough. How about $y = A\sqrt{x} + Bx^3$? Is it tasty enough for your breakfast? Actually it's fine if digested slowly:

$$\begin{aligned} y &= A\sqrt{x} + Bx^3 \\ \frac{y}{\sqrt{x}} &= A + Bx^{\frac{5}{2}} \end{aligned}$$

This looks even worse, doesn't it? But remember that it is x and y that are known. If we plot $(y/\sqrt{x})$ on the vertical, and $(x^{\frac{5}{2}})$ on the horizontal, a straight line appears, and we can read A and B from the y-intercept and gradient respectively.